Noisy Signals and Unsophisticated Players
in a Coordination Model of a Barter Economy

by

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Abstract

We apply a noisy signal approach (Morris and Shin, 1998, 2001) to the search model of a barter economy (Diamond, 1982), whose equilibria is equilibrium with and without production. Even under the existence of noisy signals, it is difficult to eliminate multiple equilibria in general because, to obtain a productive equilibrium, the individual who observes the highest value of the noisy signal must produce. Further, we investigate the role of unsophisticated traders who neglect the strategic interactions among players. We consider two types of unsophisticated players: the Bayesian traders who react to their estimates of the state of the economy given noisy signals, and the simple-minded traders who neglect the requirement that their product must be exchanged with others. The existence of the unsophisticated traders may facilitate rather than prevent the realization of a productive equilibrium. Government interventions may also have some pump-priming effects.

JEL classification: C72, C82.

Keywords: noisy signal, noisy traders, coordination, payoff externality, multiple equilibria.

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1 Introduction

In his classic article on multiple equilibria, Peter Diamond (1982) shows that, in a barter economy—which consists of two kinds of searches for production opportunities and for trading partners—multiple equilibria and unemployment develop because of a lack of coordination. Difficulties in coordination among the productive activity of individuals guarantee payoff externality and strategic complementarity for the emergence of multiple equilibria.

Various authors have written about the relationship between the problem of coordination and multiple equilibria in a macroeconomy. For example, Cooper and John (1988) demonstrated that strategic complementarity is necessary for multiple equilibria. Howitt and McAfee (1992) showed that high and low employment can be randomly switched in a job search model of workers and firms. More recently, Chamley (1999) analyzed the mechanisms of the shift between high and low activity in a model of heterogeneous production costs and imperfect information. Frankel (2003) showed that aggregated cost or productivity shocks yield a unique equilibrium in a modified model of Howitt and McAfee (1992). All these works investigate multiple equilibria or steady states caused by payoff externality of coordination in different settings. However, they hardly address the question of the relationship between multiple equilibria and the states of

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1 This paper is partly motivated by the low-growth path of the Japanese economy since 1992. One might maintain that the Japanese economy has fallen into a state of the low-level equilibrium between multiple equilibria. We see here the situation in which the level of production activity is far lower than that expected from the state of fundamentals. The Economic Planning Agency (1999) pointed out that business opportunities were reduced and the economy contracted as a whole since few people undertook new risk-taking activities. The process is figuratively similar to the following: People golf less because of pessimistic future income prospects, and golf courses close so that their employees reduce their spending, which in turn reduces the demand of golfers---.
In this paper, we study multiple equilibria in a setting where the payoff from production is explicitly related to a state of fundamentals. We use a simplified static Diamond's model\textsuperscript{2}, where the output and cost of production vary in response to the change of fundamentals of the economy.

We consider an economy with a continuum of anonymous individuals that is characterized by a state of fundamentals. Each individual chooses whether to produce consumption goods, where the number of units of consumption goods and the corresponding production costs are functions of the state of fundamentals. Following Diamond, and focusing on the critical importance of exchanges in a market economy, we assume that the produced goods cannot be consumed by the producing individual himself for certain reasons. If an individual produces consumption goods, for deriving utility from them, he needs to trade them for goods another individual produces. This causes payoff externality. The multiple equilibria in which all individuals produce (productive equilibrium) and do not produce (unproductive equilibrium) constitute the equilibrium of the economy. The appearance of not the productive equilibrium but the unproductive one is a problem common to many coordination games, as seen in the example of deer hunting in Chapter 4 of Salanié (1997)\textsuperscript{3}.

We ask if multiple equilibria can be eliminated to realize productive equilibrium as a unique equilibrium. We employ the method of Morris and Shin (1998, 2001), which is an extension of a global game of Carlsson and van Damme (1993) to the setting of a population game. Morris and Shin (1998) showed that in a currency attack model in which multiple equilibria occur in perfect information, a small amount of noise in the signal about

\textsuperscript{2} Diamond's model is originally modeled as a continuous-time one, in which the search process for production opportunities and trading partners are stochastically represented by Poisson processes, and production costs also randomly vary. These elements are simplified in our model.

\textsuperscript{3} There are \(N\) villagers indexed by \(i = 1, \ldots, N\). Each villager chooses whether to go deer hunting at his personal cost \(c_i\). The villagers can succeed in hunting and obtain payoffs above their costs only if all villagers go hunting. In this situation, an unproductive equilibrium, in which all villagers choose not to go hunting, appears.
fundamentals brings about a unique equilibrium. Their unique result is surprising, and clarifies the response of speculators to the signal that completely changes from an attacking to a not-attacking position at a critical point in the states of fundamentals.

We first show that multiple equilibria can never be eliminated in the game considered here, in which each individual considers the choices of other individuals, even though a small amount of noise is perturbed to the signal about fundamentals, as in Morris and Shin (1998, 2001). Moreover, not only the unproductive—that is, non-producing equilibrium—but also an unnatural equilibrium—such that individuals choose to produce when signals are in some interim interval but not to produce when it is in the upper region—is also permissible.

Despite the negative counterintuitive result, our first result ironically gives an important implication. The above result is due to the lack of a dominant action in the highest region of fundamentals. That is, a decision to produce is not necessarily dominant in the highest region by the structure of the model. On the other hand, in Morris and Shin (1998), it is stipulated as common knowledge that, irrespective of the choice of others, the action to attack and the action not to attack are dominant when the state of fundamentals is in the lowest and highest regions respectively. This assures the uniqueness of equilibrium, and shows that the uniqueness of equilibrium is realized by the common belief that other individuals surely choose to attack and not to attack at the extreme values of fundamentals. The mechanism is also common to the results of Frankel (2003).

Therefore, to guarantee a productive equilibrium as a unique equilibrium in our model, production must somehow be dominant in the highest region of fundamentals. Conversely, once the dominance of production is locally secured in the region, a productive equilibrium, where all individuals in a large region of fundamentals do produce, is guaranteed. That is, a productive equilibrium is comprehensively obtained only by the local dominance of production in the extreme region.

This local dominance of production is specifically represented by a positive
payoff to the production of individuals, who observe the signal in the highest states of fundamentals in our model. Then it can be obtained if it is commonly believed that the majority of individuals who observe the signal equal to unity produce. Further, we show it can be assured also by a production subsidy given only to the highest small region in the states of fundamentals. Government interventions then restore productive equilibrium.

Next, we extend our basic framework to the model that contains individuals who make unsophisticated choices as well as those who make rational choices. In our basic model, all individuals are supposed to behave in a Bayesian manner, i.e. an individual rationally chooses whether to produce based on the computation of the expected payoff to production conditional on the noisy signal about fundamentals, taking the strategic interactions between players into consideration. It appears that in the real world, however, some people behave under the strong influence of the prevailing belief or sensation for economic conditions. Moreover, other people make decisions based on their false beliefs and misperceptions, as the noise traders modeled by De Long et al. (1990).4

In the extended model, we consider an economy that consists of those unsophisticated individuals as well as rational ones. Then we show that in the case in which unsophisticated individuals make decisions based simply on the prevailing atmosphere for fundamentals, only the productive equilibrium appears under optimistic conditions, but multiple equilibria occur under pessimistic conditions. Further, in the case in which unsophisticated individuals are those who falsely and always take the signals at their face values and make decisions based only on what the signals just told them, the appearance of productive equilibrium is guaranteed independently of the conditions.

In our model a positive payoff to production in the highest states of

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4 De Long et al. (1990) modeled the noise trader as one who misperceives the expected price of the risky asset, based on the false belief that he has special information about the future price of the risky asset.
fundamentals, is never guaranteed a priori. In the extended model with
unsophisticated players, however, unsophisticated players facilitate a
positive payoff to the production of rational individuals, who observe the
signal in the highest states, and only productive equilibrium occurs. It is
interesting to note that the existence of unsophisticated players may well
serve rather than disrupt the stability of an economic system with
coordination problems. In particular, it is noteworthy in our case that the
naive contribution of unsophisticated players only in the restricted, local,
region of fundamentals is sufficient for productive equilibrium of production
in a large region of fundamentals.

The rest of this paper is organized as follows. Section 2 presents the basic
model and briefly reviews the multiple equilibria in perfect information.
Section 3 shows the result of the basic model in imperfect information, in
which the prior distribution of the state of fundamentals and the distribution
of noise are uniform. Section 4 presents the result of the extended model to
the case with noisy or unsophisticated participants, in which the prior
distribution of the state and the distribution of noise are generalized. Section
5 concludes the argument and the Appendix collects all proofs.

2 Game with Perfect Information in Basic Model

2.1 Basic Model

The essence of the coordination problem addressed in Diamond (1982) is
simply modeled by the following game in a single population (for a setting in
two populations, see section 4.7 of Vega-Redondo (1996)). We consider an
economy with a continuum of anonymous individuals, of unit mass. Each
individual chooses whether to produce $y$ units of consumption good with
production cost $c$.

When an individual produces $y$ units of consumption good, he cannot by
himself consume the goods that he produces. Moreover, for deriving utility
from them, he has to trade them for the goods another individual produces.
Therefore, an individual who has produced something is given an opportunity to trade with another individual by random matching. Then, if the individual he meets has also produced, he trades the goods with the partner and obtains payoff $c$. But if the partner has not produced, his payoff is $-c$. If an individual chooses not to produce, his payoff is fixed to be zero at the time.

Each individual is assumed to be risk neutral and to know the structure of these payoffs. Moreover, we assume that when producing is indifferent to not producing, an individual chooses not to produce. Denoting by strategy 1 and strategy 2—choosing to produce and not to produce—the above setting is represented as the population game based on the following symmetric payoff matrix:

\[
\begin{pmatrix}
  (y - c, y - c) & (-c, 0) \\
  (0, -c) & (0, 0)
\end{pmatrix}
\]

We now introduce a state of fundamentals into the model. The economy is characterized by a state of fundamentals $θ ∈ [0, 1]$, and the number of units of the produced good $y$ and the production cost $c$ are functions of $θ$. As $θ$ goes up, the fundamentals get better, and potential productivity as well as potential demand of the economy are larger. We then assume that $y(θ)$ and $c(θ)$ are continuously differentiable and satisfy the following conditions.

### A1.
$c(0) > y(0) > 0$.

### A2.
$y(1) > c(1) > 0$.

### A3.
$y'(θ) > c'(θ) > 0, \ ∀ θ ∈ [0, 1]$.

These conditions ensure that $y(θ)$ and $c(θ)$ are strictly increasing in $θ$ and there exits a unique $θ ∈ (0, 1)$ such $y(θ) = c(θ)$. Our model differs from those of Morris and Shin (1998, 2001) in that even in the highest states of fundamentals, a positive payoff to production is never assured if the proportion of individuals to produce is sufficiently small. As will be seen below, this brings about a sharp difference for the uniqueness of a solution in a game with imperfect information.

### 2.2 Game with Perfect Information
First, we consider the game with perfect information of fundamentals in which each individual knows exactly the true value of \( \theta \). When the state of fundamentals is \( \theta \) and the proportion of individuals who produce is \( l \in [0,1] \), the expected payoff to the production of an individual is \( \pi(l,\theta) = ly(\theta) - c(\theta) \).

For a given \( \theta \), denote by \( l^*(\theta) \) the proportion of individuals who produce in equilibrium. Then it holds that \( l^*(\theta) = 1 \) if \( \pi(l^*,\theta) > 0 \) and \( l^*(\theta) = 0 \) otherwise. Further, by the assumption, \( y(\theta) - c(\theta) > 0 \) if and only if \( \theta > \overline{\theta} \). Hence we have the following for the equilibria of the game:

\[
l^*(\theta) = \begin{cases} 0 & \text{if } 0 \leq \theta \leq \overline{\theta} \\ 0,1 & \text{if } \theta > \overline{\theta} \leq 1. \end{cases}
\]

Under the assumption of perfect information of fundamentals, all individuals always know the exact value of \( \theta \). Therefore, when \( 0 \leq \theta \leq \overline{\theta} \), each individual knows that \( \theta \) is in the region where the expected payoff to production is non-positive, even if all individuals produce. Then all individuals refrain from producing in the equilibrium.

On the other hand, when \( \theta < \overline{\theta} \leq 1 \), each individual knows that the expected payoff to production is positive if the proportion of individuals who produce is beyond \( \frac{c(\theta)}{y(\theta)} \), and non-positive otherwise. In this context the choice of an individual depends on his belief about the choice of other individuals. That is, if an individual believes that sufficiently many individuals choose to produce, a decision to produce is rational and otherwise a decision not to produce is rational. Thus, there occur multiple equilibria in which all individuals produce or all individuals do not produce in the region.

3 Game with Imperfect Information in Basic Model

In this section we consider a game with imperfect information in which each individual observes a noisy signal about a state \( \theta \) and, based on it, chooses whether or not to produce.
3.1 Structure and Assumption

We suppose that the state \( \theta \) is uniformly distributed on \([0, 1]\) and that when the state is \( \theta \), each individual is given a private signal \( x = S(\theta + \sigma \varepsilon) \),

\[
0 \text{ if } x < 0 \\
1 \text{ if } 1 < x
\]

where \( \sigma > 0 \) and \( S(x) = x \) if \( 0 \leq x \leq 1 \). The noise term \( \varepsilon \) is also uniformly distributed on \([-1, 1]\) and independent of \( \theta \). Further, it is independent across individuals. Here we technically remark that \( x = 0, 1 \) have a point mass and need to be treated differently from other points. We denote the cumulative distribution function of \( \theta \) by \( F \).

Each individual exactly knows about the distribution of \( \theta \) and \( x \), and the game proceeds according to the following steps:

1. Nature selects a state \( \theta \).
2. Each individual observes a private signal \( x \).
3. Based on the signal \( x \), each individual decides whether to produce.
4. Each individual carries out his decision and obtains the payoff.

We assume production function \( y(\theta) \) and cost function \( c(\theta) \) are continuously differentiable and satisfy the following conditions:

A1. \( c(0) > y(0) > 0 \).
A2'. \( y(1) > 4c(1) > 0 \).
A3'. \( y'(\theta) > 2c'(\theta) > 0 \), \( \forall \theta \in [0, 1] \).
A4'. \( y(\theta) \) is concave and \( c(\theta) \) is convex.

These conditions are met, for example, by \( y(\theta) = 4\theta, c(\theta) = \frac{1}{4}(2\theta + 1) \). In the above, A2' and A3' are to guarantee a higher payoff in the upper region than A2 and A3 in section 2.1.

For a given profile of strategies of individuals, the aggregated strategy of individuals is represented by a \([0, 1]\)-valued function \( s(x) \) on \([0, 1] \), where \( s(x) \) is the proportion of individuals who produce when the value of the signal is \( x \).
Hereafter, we always assume the measurability of \( s \).

As signals are uniformly distributed on \([\theta - \sigma, \theta + \sigma]\) when the value of the state is \( \theta \in [\sigma, 1 - \sigma] \), we have \( l(\theta, s) = \frac{1}{2\sigma} \int_{\theta - \sigma}^{\theta + \sigma} s(x) dx \). Then, for a given strategy \( s \), the payoff to an individual who produce at state \( \theta \) is

\[
\pi(l(\theta, s), \theta) = \frac{y(\theta)}{2\sigma} \int_{\theta - \sigma}^{\theta + \sigma} s(x) dx - c(\theta) \text{ for } \theta \in [\sigma, 1 - \sigma].
\]

Further, since the posterior distribution of \( \theta \) when an individual observes signal \( x \in [\sigma, 1 - \sigma] \) is the uniform distribution on \([x - \sigma, x + \sigma]\), the expected payoff to production conditional on signal \( x \) is given by

\[
u(x, s) = \frac{1}{2\sigma} \int_{x - \sigma}^{x + \sigma} \pi(l(\theta, s), \theta) d\theta = \frac{1}{2\sigma} \int_{x - \sigma}^{x + \sigma} \frac{y(\theta)}{2\sigma} \int_{\theta - \sigma}^{\theta + \sigma} s(x) dx - c(\theta)]d\theta
\]

for \( x \in [\sigma, 1 - \sigma] \). This formula is slightly modified for \( x \in (1 - \sigma, 1) \) as

\[
u(x, s) = \int_{1 - \sigma}^{x} \frac{d\theta}{1 - x + \sigma} \left[ \frac{y(\theta)}{2\sigma} \int_{\theta - \sigma}^{\theta + \sigma} s(x) dx - c(\theta)]\right],
\]

and similarly for \( x \in (0, \sigma) \).

In the case of perfect information, the choice of each individual at state \( \theta \) depends only on that of other individuals at \( \theta \), and it is independent of those at other states. On the other hand, in the case of imperfect information, the choice of each individual who observes signal \( x \) depends on that of other individuals whose signals are in \([x - 2\sigma, x + 2\sigma]\) through the above computation of \( u(x, s) \).

We now state some definitions about the equilibria of the game.

**Definition 1:** The (aggregated) strategy \( s \) is optimal in the game if and only if \( s(x) = 1 \) whenever \( u(x, s) > 0 \) and \( s(x) = 0 \) whenever \( u(x, s) \leq 0 \).

**Definition 2:** An equilibrium of the game is a profile of strategies of individuals that represents an optimal aggregate strategy.
When $s$ is optimal, $s$ takes values in $\{0,1\}$. Clearly $s=0$ is always an optimal strategy, and the corresponding equilibrium is called unproductive:

**Definition 3:** An unproductive equilibrium is a profile in which each individual always chooses not to produce.

In contrast, an equilibrium that represents an optimal strategy, of which form is of $1,(1-k)$ for some $k \in (0,1)$, is called productive.

**Definition 4:** A productive equilibrium is a profile that represents an optimal strategy in which each individual chooses to produce if and only if his private signal is above some $k \in (0,1)$.

As will be seen later, in general not only the unproductive equilibrium but an unnatural equilibrium such that $s(x_1) = 1$ and $s(x_2) = 0$ for some $(x_1, x_2)$ with $x_1 < x_2$ is not excluded. To exclude these equilibria, we impose the following boundary condition on $s$, which substitutes for the assumption of the upper dominance region in Morris and Shin (1998, 2001). Here it may be said that this local condition is slightly weaker than the assumption of the upper dominance region because it follows from the assumption.

**(RC: Regular Condition)** $s(1) = 1$.

Condition (RC) is satisfied for any sufficiently small $\sigma > 0$ in a self-fulfilling way if it is commonly believed that $s(l) > \frac{3c(l)}{y(l)}$. Because $u(l,s) > 0$, which means $s(l) = 1$, always holds for any sufficiently small $\sigma > 0$ if all individuals commonly believe that strictly more than $\frac{3c(l)}{y(l)}$ of individuals whose signal is one always choose to produce. In the below, by “under condition (RC),” we mean “among all strategies that satisfy $s(l) = 1$."

### 3.2 Results

For the statement of the following proposition, we denote by $l(\theta, \sigma)$ the proportion of individuals who end up producing at the state $\theta$. Further, we
define social welfare $W(\theta, \sigma)$ and its total value $\overline{W}(\sigma)$ by $W(\theta, \sigma) = l(\theta, \sigma)(y(\theta)/l(\theta, \sigma) - c(\theta))$ and $\overline{W}(\sigma) = \int_0^l W(\theta, \sigma)d\theta$ respectively.

**Proposition 1**: Under condition (RC), for any $\sigma \in (0, \sigma_0)$ where $\sigma_0 =\lbrack \sup\{\sigma > 0 | y(1-2\sigma) > c(1)\} \rbrack \wedge \frac{\sigma}{2}$, there exists a unique equilibrium such that all individuals whose signal $x \leq k(\sigma)$ decide not to produce, and all individuals whose signal $x > k(\sigma)$ decide to produce for some $k(\sigma) \in (0,1)$.

Consequently, the following hold for $l$ and $W$ in the outcome, and we have

$$\overline{W} = \int_{k-\sigma}^{k+\sigma} \frac{1}{2\sigma}(\theta + \sigma - k)(y(\theta)/(\theta + \sigma - k) - c(\theta))d\theta + \int_{k-\sigma}^l (y(\theta) - c(\theta))d\theta .$$

(i) $l = 0$ and $W = 0$ if $0 \leq \theta \leq k - \sigma$.

(ii) $l = \frac{1}{2\sigma} (\theta + \sigma - k)$ and $W = l(\theta, \sigma)(y(\theta)/l(\theta, \sigma) - c(\theta))$ if $k - \sigma < \theta < k + \sigma$.

Moreover, for any sufficiently small $\sigma$, there exists some $\theta^* \in (k - \sigma, k + \sigma)$ such that $W < 0$ if $k - \sigma < \theta < \theta^*$ and $W > 0$ if $\theta^* < \theta < k + \sigma$.

(iii) $l = 1$ and $W = y(\theta) - c(\theta) > 0$ if $k + \sigma \leq \theta \leq 1$.

**Explanation**: As already mentioned, in general we can not eliminate unproductive equilibrium. However, the first part of Proposition 1 implies that the regular condition assures that the productive equilibrium is a unique solution. That is, under the situation that all individuals whose signal is unity always choose to produce and it is commonly believed, the productive equilibrium is a unique equilibrium. This result is explained as follows.

Suppose that an individual considers whether to produce when he observes signal $x_1 \in (\theta, 1)$. In this case he calculates the conditional distribution of signals conditioned on his private signal $x_1$, and then estimates the expected
payoff to production with the responses of other individuals to signal $x \in [x_i - 2\sigma, x_i + 2\sigma]$. So, he has to expect the responses of other individuals to signal $x \in [x_i - 2\sigma, x_i + 2\sigma]$ for that. But they depend on the responses to signal $x \in [x_i - 4\sigma, x_i + 4\sigma]$, and so on. Thus, he reaches to expect the responses of other individuals to signal $x \in [\theta - 2\sigma, \theta]$, since it is commonly known that individuals whose signal is below $\theta - 2\sigma$ surely choose not to produce.

At this stage, an optimal strategy for individuals is uniquely obtained by a “backward induction” when the boundary condition $s(1) = 1$ is given. To begin with, positive $u(x, s)$ in a neighborhood of 1, say $(1 - \varepsilon, 1)$, is assured by $s(1) = 1$. In turn this ensures positive $u(x, s)$ in $(1 - \varepsilon, 1]$ for some small $\varepsilon > 0$. By continuing this procedure, the turning point $k(\sigma)$, where individuals whose signal $x > k(\sigma)$ choose to produce, and individuals whose signal $x \leq k(\sigma)$ choose not to produce, is uniquely determined and $s = 1_{\{k(\sigma), 1\}}$ is obtained as a unique optimal strategy for individuals, since $y - c$ is continuous, strictly increasing and non-positive on $[0, \theta - 2\sigma]$. This is why only the productive equilibrium, where all individuals in the large region of fundamentals do produce, occurs as a snow slide takes place, once it is commonly believed that all individuals whose signal is unity always choose to produce.

(ii) In the second part of proposition 1 ensures that when $\theta$, the state of fundamentals, is within $(k(\sigma) - \sigma, \theta^*)$, a proportion, $\frac{1}{2\sigma}(\theta + \sigma - k(\sigma))$, of individuals will end up producing and suffer losses. This superficially seems paradoxical to the notion that each individual optimizes his choice to the private signal. Why does this occur?

As later seen in the proof, when $k(\sigma) - \sigma < \theta < \theta^*$, $l(\theta, I_{(k(\sigma), 1)}) < c(\theta) \frac{\theta}{y(\theta)}$ and then $\pi(l, \theta) < 0$. That is, the proportion $l(\theta, I_{(k(\sigma), 1)})$ of individuals who observe the signal $x$ beyond $k(\sigma)$ and choose to produce is so small that the expected payoff to production is negative. But their choices to produce
are rational in the Bayesian sense, because the expected payoff to the production of the individual is positive, which easily follows from that $u(x, I_{k(\sigma)})$ is strictly increasing for $k(\sigma) - \sigma \leq x \leq k(\sigma) + \sigma$ and $u(k, I_k) = 0$.

The limit value of $k(\sigma)$ when $\sigma$ goes to zero is easily obtained as a corollary by taking the limit as $\sigma \to 0$ in

$$u(k, I_{(k, \theta)}) = \int_{\theta}^{k} \frac{dz}{2} \left[ \frac{1}{2} y(k - \sigma z) - c(k - \sigma z) \right] = 0.$$ 

Thus we can state the next corollary:

**Corollary 1:** $\lim_{\sigma \to 0} k(\sigma)$ is the solution of $\frac{1}{2} y(x) - c(x) = 0$.

**Remarks:** Corollary 1 implies $k(\sigma) > 0$ for any sufficiently small $\sigma > 0$. This shows that even the productive equilibrium is inefficient in imperfect information. Because with imperfect information each individual whose signal $x$ is in $(\theta, k)$ never produces, although with perfect information the state that all individuals produce is possible when state $\theta$ is in $(\theta, k)$.

In intuition, Corollary 1 is due to the fact that when the value of his private signal is $k$ and all individuals behave based on $I_{(k, \theta)}$, he expects that half of individuals choose to produce, since the prior distribution is uniform, as pointed out in Morris and Shin (2001).

If condition (RC), which substitutes for the assumption of the upper dominance region in Morris and Shin (1998, 2001), always holds, only the productive equilibrium appears by proposition 1. This implies that there is room for a policy treatment to guarantee good equilibrium by ensuring condition (RC). Proposition 2, which immediately follows as a corollary of proposition 1 from
Proposition 2: Suppose it is commonly known that when the state \( \theta \) is in \((1-\sigma,1]\), any individual who produces is given the subsidy of \((c(\theta)+\lambda)\) for any \(\lambda > 0\). Then \(s = l_{(1-\sigma,1]}\) is a unique optimal strategy for all the individuals.

Explanation: Proposition 2 tells us that it suffices that a subsidy is given only when state \( \theta \) is in the highest region \((1-\sigma,1]\), where \(\sigma\) is the width of the error term. Also, \(\lambda\) can be a very small number. That is, the announcement of making up the production cost in the highest small region ensures the condition (RC) that individuals who observe the signal equal to unity surely choose to produce. Then, this guarantees the appearance of only the productive equilibrium, where all individuals in a large region of fundamentals do produce, by backward induction through individuals as already stated. The important point is that the announcement of a policy treatment is required only for the restricted region of fundamentals to obtain the productive equilibrium comprehensively.

This proposition implies that as long as the government provides the small group of individuals who obtain signals near the top with a little more than the production cost, then productive equilibrium will emerge. This can be done only if the government can verify the signal and only if the government can collect tax revenues by lump-sum taxes, of which the regressive property might be discussed in terms of the fairness of taxation.

If the signal is not verifiable even ex post, then the government should engage in production to ensure the condition (RC) if it can know the true value of \( \theta \), or the government should hire some individuals who will produce when the signal is unity. Here the publicly hired individuals must be honest and must to be believed to be honest. Here, we see a resurrection of J. M. Keynes' idea of "pump-priming expenditure theory," because the required amount of subsidies for production can be rather small. As will be indicated
in the following example, this subsidy policy seems to be fairly effective as a pump-priming mechanism for production.


The next proposition specifically shows that without condition (RC) not only \( s \equiv 0 \) but also another unnatural solution is allowed in the case where \( y \) and \( c \) are linear.

Proposition 3: Suppose that \( y \) and \( c \) are linear functions such that

\[
0 < \frac{c'}{y} < \frac{5}{24} \quad \text{and} \quad \inf_{\theta \in (0,1)} \frac{2c(\theta)}{y(\theta)} < \varphi(-2 + \sqrt{9 - 24c'/y'}) , \quad \text{where} \quad \varphi(x) = \frac{3}{4} - \frac{1}{2}x - \frac{1}{4}x^2 .
\]

Then, there exist \( k_1, k_2 \) \((0 < k_1 < k_2 < 1)\) such that \( s = 1_{(k_1, k_2)} \) is an optimal strategy for a sufficiently small \( \sigma > 0 \).

The following numerical example indicates that \( k(\sigma) \) is much larger than \( \theta \), which means that even the productive equilibrium is inefficient. This is because the region of fundamentals, where a positive payoff to production is possible but all individuals choose not to produce, is not small, even in the productive equilibrium. Further, we can see how effective the subsidy in Proposition 2 is as a pump-priming policy.

Example 1: For \( y(\theta) = 4\theta, c(\theta) = \frac{1}{2}\theta + \frac{1}{4}, \theta = \frac{1}{14} = 0.07, \sigma_0 = \frac{1}{28} = 0.04 \). Then, by taking \( \sigma = \frac{1}{32} = 0.03 \), we have \( k(\sigma) = \frac{11}{72} = 0.15 \) and \( \bar{W} = 1.49 \) for Proposition 1.

Denoting the amount of total subsidies by \( \bar{S} \), we have \( \bar{S} = 0.02 \) and \( \frac{\bar{W}}{\bar{S}} = 64.27 \) for \( \sigma = \frac{1}{32} = 0.03 \) and \( \lambda = 0 \) for Proposition 2.

Further, it holds for Proposition 3 that

\[
\inf_{\theta \in (0,1)} \frac{2c(\theta)}{y(\theta)} = \frac{3}{8} = 0.34 , \quad \varphi(-2 + \sqrt{6}) = -\frac{3}{4} + \frac{1}{2}\sqrt{6} = 0.47 , \quad k_1 = \frac{1}{4\sqrt{6} - 8} = 0.56
\]
and \( k_2 - k_1 = (1 - A')\sigma = (3 - \sqrt{6})\sigma = 0.02 \).

### 4 Game Under Imperfect Information with Unsophisticated Players

In this section we extend the model in two ways. One extension is to generalize the distribution of \( \theta \) and \( \epsilon \), which enables us to fit our model to a broader range of situations. In particular, by changing the distribution of \( \theta \) we can represent various situations in which the priors for the state of fundamentals, i.e., the prevailing sensations for the fundamentals, are different. It is useful to analyze the impact of the prior on the equilibrium of the game.

The other is to introduce two types of individuals who choose whether to produce without considering the choice of other individuals. It is up to the reader if she calls these players more “realistic”. In the basic model all individuals are identical and optimize the choice in a Bayesian manner, in which they compute the expected payoff to production based on private signals, taking the choices of others into account. We call these individuals type B (Bayesian). As an extension of the basic model, first we consider the situation in which a proportion of individuals make decisions simply based on the prevailing sensation for the state of fundamentals. This characterization implies that the individuals are not analytic, but naive, under the strong influence of the atmosphere prevailing in the economy. We call that type of individual type S (Sensation).

Next, we consider the case in which a different kind of unsophisticated individual, who we call type N (Noise trader), coexists with type B individuals. Type N people falsely and always take the signals at their face values and make decisions based only on what the signals just told them. They never take account the strategic interactions of players like type S, and they neglect any prior unlike type S because they completely believe that
they know the true value of $\theta$ based on the signals, somewhat like the noise traders in De Long et al. (1990).

In the previous section, we have shown that if all individuals are Bayesian, generally multiple equilibria may emerge, and productive equilibrium as well as unproductive equilibrium may appear. In this section we show that in the case in which type S-individuals are introduced, if the prevailing atmosphere for fundamentals is optimistic, only productive equilibrium appears, as long as the proportion of type S is not so small. Further, in the case in which type N individuals are introduced, the appearance of productive equilibrium is guaranteed independently of the atmosphere as long as the proportion of type N stays within some range.

In any case, the players who play the Bayesian equilibrium game know what unsophisticated players do. It is interesting to note that the existence of unsophisticated players does not necessarily make the situation more difficult. In our cases, unsophisticated players facilitate the fulfillment of condition (RC) for rational players. Then this fulfillment of the local condition at the extreme value of fundamentals brings about only the productive equilibrium of production in a large region. Thus bounded rationality, if not ignorance, may well serve the stability of an economic system with coordination problems.

4.1 Structure and Assumption

Now we suppose that the state $\theta$ is a random variable on $[0,1]$, of which distribution function is denoted by $G$, which has a density $g$ that is strictly positive and continuously differentiable. We denote by $\theta_m^G$ the expectation value of $\theta$ by $G$. The noise term $\varepsilon$ is distributed on $[-1,1]$ with distribution function $F$, which also has a continuous density $f$ and of which support is assumed to be not contained in $[-1,0]$. As in the previous section, $\varepsilon$ is independent of $\theta$ and independent across individuals. Further, there is some critical value $\theta_r \in (0,1)$ such that the state of fundamentals is regarded as productive (unproductive) when the true value

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of $\theta$ exceeds (resp. does not exceed) $\theta_r$. The prevailing sensation for economic conditions is reflected on the prior $G$, and we say that the atmosphere is optimistic (pessimistic) when $\theta^G$ exceeds (resp. does not exceed) $\theta_r$.

We consider two cases in which sophisticated and unsophisticated individuals coexist in the economy: Case 1 assumes that type B (Bayesian) and type S (Sensational) coexist, while Case 2 assumes that type B and type N (Noise trader) coexist. In both cases we assume that the ratio of type B is $\rho$ and that of type S and type N is $1 - \rho$ for $\rho \in (0,1]$.

In Case 1, individuals of type S decide whether to produce depending only on the prevailing atmosphere, i.e., they choose to produce if the atmosphere is optimistic, and otherwise choose not to produce. Meanwhile, in Case 2, individuals of type N do it based only on what their signals just told them, i.e., those who observe the signal $x$, falsely believe that the true value of $\theta$ is $x$, and they choose to produce if $x > \theta_r$ and otherwise choose not to produce. On the other hand, individuals of type B choose to produce based on the Bayesian computation of the expected payoff, knowing how unsophisticated individuals make decisions and taking into account the strategic interactions of players. In the above, individuals of type B know that $G,F,\theta$, and $\rho$, but individuals of type S and type N know only $G$ and $\theta_r$.

In the extended model the game proceeds according to the following steps:

1. Nature selects a state $\theta$ according to $G$.
2. Each individual is given a private signal $x$.
3. In Case 1, an individual of type S whose signal is $x$ chooses to produce if the atmosphere is optimistic and otherwise chooses not to produce, irrespective of the signal $x$. In Case 2, an individual of type N whose signal is $x$ chooses to produce if $x > \theta_r$ and otherwise chooses not to produce. In the both cases an individual of type B whose signal is $x$ decides whether to produce based on the expected payoff conditional on $x$, knowing how
individuals of type S and N make decisions and considering the strategic interactions of players.

4. Each individual carries out his decision and obtains the payoff.

The choice of individuals of type S (type N) are exogenously determined by $G$ and $\theta_r$ (resp. $F,G$ and $\theta_r$), and only individuals of type B are the Bayesian players of the game. Knowing this, individuals of type B choose whether to produce based on the signal $x$. The aggregated strategy of type B is represented by a $[0,1]$-valued measurable function $s$ as in the basic model, and an equilibrium of the game in the extended model is also defined as in the basic model.

Definition 3: An unproductive equilibrium is a profile in which each individual of type B always chooses not to produce.

Definition 4: A productive equilibrium is a profile that represents an optimal strategy, in which each individual of type B chooses to produce if and only if his private signal is above some $k \in (0,1)$.

Production function $y(\theta)$ and cost function $c(\theta)$ are continuously differentiable and satisfy the following conditions, which are also met by $y(\theta) = 4\theta, c(\theta) = \frac{1}{4}(2\theta + 1)$ if $F$ is uniform. As will be seen in Proposition 4 and Proposition 5, $\rho$ gives the lower bound of the proportion of type B for the existence of the productive equilibrium to be ensured in the case where individuals of type S or type N choose not to produce. Meanwhile, $1 - \rho$ gives that of type S and type N for the productive equilibrium to appear as a unique equilibrium in the case where individuals of type S or type N choose to produce.

A1. $c(0) > y(0) > 0$.

A2”. $y(l) > \frac{2c(l)}{(1 - F(0))} > 0$.

A3’. $y'(\theta) > 2c'(\theta) > 0, \forall \theta \in [0,1]$.
A4". \( p < \bar{p} \), where \( \bar{p} = (\max_{0 \leq \theta \leq 1} 2c'(\theta)) \vee \left( \frac{2c(1)}{y(1)(1 - F(0))} \right) \) and \( \bar{p} = 1 - \frac{c(1)}{y(1)} \).

### 4.2 Results

Here we present results for Cases 1 and 2. In both cases, multiple equilibria can be eliminated for productive equilibrium since unsophisticated players (individuals of type S and type N) facilitate the fulfillment of local condition (RC) for rational players (individuals of type B). We start with the result for Case 1, where individuals of type B and type S coexist.

**Proposition 4**: In the case where individuals of type B and type S coexist, when the atmosphere is optimistic (pessimistic), for any \( \rho \in (0, 1] \) (resp. \( \rho \in (\underline{\rho}, 1] \)) and any sufficiently small \( \sigma > 0 \), there exists \( k(1, \sigma, \rho) \) (resp. \( k(0, \sigma, \rho) \)) such that \( s = l(1, \sigma, \rho, \underline{\rho}, 1] \) (resp. \( s = l(0, \sigma, \rho, \underline{\rho}, 1] \)) gives the productive equilibrium of the game.

In particular, \( s = l(1, \sigma, \rho, \underline{\rho}, 1] \) gives a unique equilibrium if and only if the atmosphere is optimistic and \( 0 < \rho < \bar{p} \). While \( s = 0 \) gives also an equilibrium as well as \( s = l(0, \sigma, \rho, \underline{\rho}, 1] \) if the atmosphere is pessimistic and \( \underline{\rho} < \rho \leq 1 \).

**Remarks**: Proposition 4 implies the following. Under both pessimistic and optimistic atmospheres, if the proportion of type B is sufficiently large, productive equilibrium—where all individuals of type B whose signal is above some threshold value choose to produce—may exist.

However, when the elimination of multiple equilibria for the productive equilibrium is guaranteed, it is restricted to the case in which the atmosphere is optimistic and the proportion of type N is above \( 1 - \bar{p} \). Therefore, this seems more likely to happen, for instance in the situation in
which individuals of type B become type S and choose to produce without considering the choices of other individuals in an optimistic atmosphere.

For obtaining only productive equilibrium in other cases, some policy treatment is needed like the production subsidy explained in the previous section.

The next corollary is proved as Corollary 1 and useful for the computation of \( \lim_{\sigma \to 0} k(1, \sigma, \rho) \) and \( \lim_{\sigma \to 0} k(0, \sigma, \rho) \).

**Corollary 2:** \( \lim_{\sigma \to 0} k(1, \sigma, \rho) \) (\( \lim_{\sigma \to 0} k(0, \sigma, \rho) \)) is given by the solution of \( (1 - \frac{\rho}{2}) y(x) - c(x) = 0 \) (resp. \( \frac{\rho}{2} y(x) - c(x) = 0 \)).

Next, we turn to the result for Case 2, in which individuals of type B and type S coexist. In this case an individual of type B knows that all individuals of type N choose to produce when his signal \( x \) is more than \( \theta_r + 2\sigma \) and choose not to produce when his signal \( x \) is less than \( \theta_r - 2\sigma \). Based on this fact, Proposition 5 is also proved by making use of lemmas ready for Proposition 4 and stated in terms of \( k(1, \sigma, \rho) \) and \( k(0, \sigma, \rho) \) in Proposition 4.

**Proposition 5:** In the case where individuals of type B and type N coexist, for any sufficiently small \( \sigma > 0 \) and any \((\rho, \theta_r) \in (\rho, 1] \times (0, 1)\) such that either \( \theta_r < \lim_{\sigma \to 0} k(1, \sigma, \rho) \) or \( \lim_{\sigma \to 0} k(0, \sigma, \rho) < \theta_r \) holds, there exists \( k(\sigma, \rho) \) with \( k(1, \sigma, \rho) < k(\sigma, \rho) < k(0, \sigma, \rho) \) such that \( s = l_{k(\sigma, \rho), l1} \) gives the productive equilibrium. In particular, \( s = l_{k(\sigma, \rho), l1} \) gives a unique equilibrium if such \( \rho \) is contained in \((\rho, \rho)\).
Remarks: It seems that $\lim_{\sigma \to 0} k(0, \sigma, \rho) < \theta_\tau$ usually holds, as will be indicated in the following example. Therefore, roughly speaking, the proposition says that the elimination of multiple equilibria for productive equilibrium is guaranteed only by the condition that the ratio of type N stays within $(1-\varphi, 1-\rho)$. An optimistic atmosphere is no longer needed, in contrast to Case 1. Because when an individual of type B observes the signal equal to unity, he knows that all individuals of type N surely choose to produce, which implies condition (RC).

Therefore, it is easy to see for the same reason that Proposition 5 holds even for the case where individuals of type N make decisions based on the posterior distribution of $\theta$ conditional on the signals as long as they neglect the strategic interactions of individuals, although in the present case individuals of type N simply always take the signals always at their face values.

Finally, we give an example for Cases 1 and 2. The example indicates that $(\rho, \varphi)$, the period of the proportion of type B where only the productive equilibrium can appear in Case 2, is rather restricted. On the other hand, in Case 1 the atmosphere must be optimistic for obtaining only productive equilibrium. Therefore, the elimination of multiple equilibria for productive equilibrium does not seem so easy in general despite the presence of type S and N.

Example 2: For $y(\theta) = 4\theta, c(\theta) = \frac{1}{4}(2\theta + 1), dF(z) = \frac{1}{2} d\theta, \theta_\tau = \frac{1}{2}$, it holds that $\rho = \frac{1}{4} \sqrt{\frac{3}{4}} = \frac{3}{4}$ and $\varphi = \frac{13}{16} = 0.81$. By taking $\rho = \frac{25}{32}$ and $\sigma \to 0$, we have $\lim_{\sigma \to 0} k(0, \sigma, \rho) = \frac{4}{17} = 0.24$, $\lim_{\sigma \to 0} k(1, \sigma, \rho) = \frac{4}{31} = 0.13$, which shows
\[
\lim_{t \to 0^+} k(0, \sigma, \rho) < \theta_f.
\]

Hence by Proposition 4, for any \( G \) such that

\[
\theta_m > \frac{1}{2} \text{ as } g(\theta) = \frac{1}{K} \exp\left(-\frac{(\theta - 3/5)^2}{200}\right) \text{ with } K = \int_0^1 \exp\left(-\frac{(\theta - 3/5)^2}{200}\right) d\theta,
\]

the limit of the productive equilibrium \( s \) is given by \( s = 1_{\left(\frac{4}{31}\right)} \), which gives the limit of a unique equilibrium for \( G \) and \( \rho = \frac{25}{32} \). But, if \( G \) is such that \( \theta_m \leq \frac{1}{2} \) as \( g(\theta) = 1 \), the set of the limit of the equilibria contains the unproductive equilibrium as well as the productive one given by \( s = 1_{\left(\frac{4}{31}\right)} \).

Further, by Proposition 5, for any sufficiently small \( \sigma > 0 \), there exists \( k(\sigma, \rho) \) with \( k(1, \sigma, \rho) < k(\sigma, \rho) < k(0, \sigma, \rho) \) such that \( s = 1_{\left(\frac{4}{31}\right)} \) gives a productive and unique equilibrium for Case 2.

### 5 Concluding Remarks

In this paper, we have asked the question how multiple equilibria can be eliminated in a simplified barter economy model with imperfect information. We have found that in general multiple equilibria cannot be eliminated. A sufficient condition for the elimination of multiple equilibria and realization of the equilibrium with production is that each individual, who observes the signal about the state of the highest value of fundamentals, never fails to produce. Furthermore, we have shown that multiple equilibria can be eliminated for productive equilibrium in the presence of unsophisticated individuals. Curiously, it seems to indicate the possibility that bounded rationality may well serve rather than disrupt the stability of an economic system that possesses coordination problems.
Our results have the following implication to the nature of multiple equilibria in a macroeconomy. Equilibria are represented as response functions of individuals to the noisy signal about the state of fundamentals. Productive equilibrium implies that all rational individuals produce in the upper region exceeding a critical level of the states of fundamentals. This feature is brought about by the application of the method of Morris and Shin (1998, 2001).

These results in a standard model are based on two fundamental assumptions of the models. One is the assumption of imperfect information, i.e., all individuals do not know the true value of the state of fundamentals and are given the noisy private signals about the state. The other is that all the rational players behave in the Bayesian manner, i.e., they optimize their choices conditional on the signal, while unsophisticated individuals make decisions in irrational ways. These two assumptions make the response to one signal interdependent with that to another signal, and equilibria are obtained as response functions. This mechanism never disappears however infinitesimal the amount of noise is, nor if one introduces, not strictly the Bayesian, unsophisticated players into the model. The coexistence of unsophisticated players, however, may mitigate some difficulties.

A Proofs

A.1 Proofs for Section 3

The proof proceeds in the same way as in Morris and Shin (1998). For a measurable set $B \subset [0,1]$, we denote by $I_s$ the indicator function of $B$. Then we have the following lemmas.

**Lemma 1:** $u(k, I_{(k,d)})$ is continuous on $(0,1)$ and has a unique zero point
\( k(\sigma) \) in \((0,1)\) for any \( \sigma \in (0,\sigma_0) \), where \( \sigma_0 = [\sup\{\sigma > 0 \mid y(1 - 2\sigma) > c(1)\}] \cdot \frac{\theta}{2} \).

**Proof of Lemma 1.** By the definition of signal \( x \), \( u(k, I_{(k,1)}) \) is obviously continuous on \((0,1)\). For the proof we show that \( u(k, I_{(k,1)}) \) is strictly increasing on \((\sigma, 1 - \sigma)\) and that \( u(k, I_{(k,1)}) < 0 \) for \( k \in [0, \sigma] \) and \( u(k, I_{(k,1)}) > 0 \) for \( k \in [1 - \sigma, 1) \).

First, by using the concavity and convexity of \( y, c \), for \( k \in (\sigma, 1 - \sigma) \) we have

\[
\frac{d}{dk} u(k, I_{(k,1)}) = \frac{1}{2\sigma} \left[ y(k + \sigma) - \frac{1}{2\sigma} \int_{k-\sigma}^{k+\sigma} y(\theta) d\theta - (c(k + \sigma) - c(k - \sigma)) \right]
\]

\[
\geq \frac{1}{2\sigma} \left[ y(k + \sigma) - y(k) - 2(c(k + \sigma) - c(k)) \right]
\]

\[
= \frac{1}{2\sigma} \int_{k-\sigma}^{k+\sigma} [y'(\theta) - 2c'(\theta)] d\theta > 0,
\]

which shows \( u(k, I_k) \) is strictly increasing on \((\sigma, 1 - \sigma)\).

Next, since \( \pi(l, \theta) < 0 \) for all \((l, \theta) \in [0,1] \times [0, \theta] \) by the assumption, clearly \( u(k, I_{(k,1)}) < 0 \) for \( k \in [0, \sigma] \). As for \( k \in [1 - \sigma, 1) \)

\[
u(k) = \int_{k-\sigma}^{k+\sigma} \pi(1 - F) \left( \frac{k-\theta}{\sigma} \right) d\theta = \int_{k-\sigma}^{k+\sigma} \pi(1 - F(z), k - \sigma z) \frac{\sigma dz}{1 - k + \sigma}
\]

\[
> y(k - \sigma) \int_{k-\sigma}^{k+\sigma} \left( 1 - F(z) \right) \frac{\sigma dz}{1 - k + \sigma} - c(1).
\]

Changing variables to \( \frac{k-1}{\sigma} = \eta (1 \leq \eta < 0) \), we have

\[
\int_{\frac{k-1}{\sigma}}^{\frac{k+1}{\sigma}} (1 - F(z)) \frac{\sigma dz}{1 - k + \sigma} = \int_{\eta}^{1-\eta} (1 - F(z)) \frac{\sigma dz}{\sigma(1-\eta)} = \frac{1}{1-\eta} \int_{\eta}^{1-\eta} \frac{1-z}{2} dz = \frac{1-\eta}{4} > \frac{1}{4}.
\]
Thus, for $1 - \sigma \leq k < 1$, we have

$$u(k, k) > \frac{y(k - \sigma) - c(1)}{4} \geq \frac{y(1 - 2\sigma) - c(1)}{4} \geq 0$$

if $0 < \sigma \leq \sigma_0$, where $\sigma_0 = \sup\{\sigma > 0 \mid y(1 - 2\sigma) > c(1)\}$.

**Lemma 2**: Under condition (RC), $s = I_{(k, 1)}$ is a unique optimal strategy for individuals.

**Proof of lemma 2**. For any optimal strategy $s$ for individuals, set $\bar{x}_s = \sup\{0 < x < 1 \mid s(x) < 1\}$ and $\underline{x}_s = \inf\{0 < x < 1 \mid 0 < s(x)\}$. It is easily shown that $\bar{x}_s \geq \underline{x}_s$.

For $x \in [1 - \sigma, 1)$, we have

$$\lim_{x \to 0^+} u(x, s) = \lim_{x \to 0^+} \int_{x - \sigma}^1 \frac{d\theta}{1 - x + \sigma} \int_{\theta - \sigma}^\theta s(t) \, dt + \frac{\theta + \sigma - 1}{2\sigma} \, s(1) \, d\theta$$

$$\geq \int_{x - \sigma}^1 \frac{d\theta}{\sigma} \left[ \frac{\theta + \sigma - 1}{2\sigma} \right] \, s(1) \, d\theta$$

$$= \frac{1}{4} \frac{y(1 - \sigma) - c(1)}{2\sigma} > 0,$$

which implies $\bar{x}_s < 1$.

When $s(x) < 1$, there are some individuals who do not produce, which means $u(x, s) \leq 0$. From this we have $\lim_{x \to \tau} u(x, s) \leq 0$.

Clearly $\lim_{x \to \tau} u(x, s)$ is increasing in $s$, i.e., $\lim_{x \to \tau} u(x, s_1) \geq \lim_{x \to \tau} u(x, s_2)$ for any pair of measurable functions such that $s_1(\theta) \geq s_2(\theta), \forall \theta \in [0, 1]$. So, by $I_{(1, \tau)}(\theta) \leq s(\theta), \forall \theta \in [0, 1]$ we have $\lim_{x \to \tau} u(x, I_{(1, \tau)}) \leq \lim_{x \to \tau} u(x, s) \leq 0$, which shows $\bar{x}_s \leq k(\sigma)$.  

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By the symmetric argument, we have \( \underline{x} \geq k(\sigma) \). Therefore, \( \overline{x} = \underline{x} = k(\sigma) \) and \( s = I_{(k(\sigma), 1]} \).

**Proof of proposition 1.** Since \( l(\theta, I_{(k(\sigma), 1]}) = \frac{1}{2\sigma} \int_{\theta - \sigma}^{\theta + \sigma} I_{(k(\sigma), 1]}(\theta) d\theta \) and \( \pi(\theta) = l_y(\theta) - c(\theta) \), (i) and (iii) immediately follow. By assumption, there exist \( \beta_1, \beta_2 \in (0,1) \) such that \( \beta_1 \leq \frac{c(\theta)}{y(\theta)} \leq \beta_2, k(\sigma) - \sigma \leq \theta \leq k(\sigma) + \sigma \). So by the continuity of \( l(\theta, I_{(k(\sigma), 1]}) \) and \( \frac{c(\theta)}{y(\theta)} \), \( l(\theta, I_{(k(\sigma), 1]}) \) crosses \( \frac{c(\theta)}{y(\theta)} \) from below only once at \( \theta^* \in (k(\sigma) - \sigma, k(\sigma) + \sigma) \), which implies (ii).

**Proof of proposition 3.** For \( k_i < k_j \leq 1 - 2\sigma \) with \( k_j - k_i \leq 2\sigma \), we have

\[
u(k_i, I_{(k_i, k_j)}) = \frac{1}{2\sigma} \left[ \int_{k_i - \sigma}^{k_i + \sigma} d\theta \left[ \frac{\theta + \sigma - k_i}{2\sigma} y(\theta) - c(\theta) \right] + \int_{k_i + \sigma}^{k_i + \sigma} d\theta \left[ \frac{k_j - k_i}{2\sigma} y(\theta) - c(\theta) \right] \right]
\]

\[
= \frac{1}{2} \int_\lambda^\rho d\lambda [\frac{1-z}{2} y(k_i - \sigma \lambda) - c(k_i - \sigma \lambda)] (z = \frac{k_j - \theta}{\sigma})
\]

\[
+ \frac{1}{2} \int_\lambda^\rho d\lambda [\frac{1-A}{2} y(k_i - \sigma \lambda) - c(k_i - \sigma \lambda)].
\]

where \( \lambda = 1 + \frac{k_j - k_i}{\sigma} \). Similarly

\[
u(k_j, I_{(k_i, k_j)}) = \int_{k_i - \sigma}^{k_i + \sigma} d\lambda [\frac{1-A}{2} y(k_i - \sigma \lambda) - c(k_i - \sigma \lambda)]
\]

\[
+ \frac{1}{2} \int_\lambda^\rho d\lambda [\frac{1-z}{2} y(k_j + \sigma \lambda) - c(k_j + \sigma \lambda)].
\]

Then, for linear functions \( y, c \), we have

\[
u(k_j, I_{(k_i, k_j)}) - \nu(k_i, I_{(k_i, k_j)})
\]

\[
= \frac{1}{2} \int_\lambda^\rho d\lambda [\frac{1-z}{2} y(1) - c(1)][\sigma (1-A) + 2\sigma \lambda]
\]

\[
= (1-A)[-\frac{1}{12} y(1)A^2 - \frac{1}{3} y(1)A + \left(\frac{5}{12} y(1) - 2c(1)\right)] \sigma
\]

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Thus, \( u(k_2, I(k_1, k_2)) - u(k_1, I(k_1, k_2)) = 0 \) has solution \( A^* = -2 + \sqrt{9 - 24c' / y'}(l) \) in 
(0,1) when \( \frac{e^x}{y} < \frac{5}{24} \).

Next, we observe that \( u(k_1, I(k_1, k_2)) \) converges to

\[
\frac{1}{2} \int_\Delta dz \left[ \frac{1 - z}{2} y(k_1) - c(k_1) \right] + \frac{1}{2} \int_\Delta dz \left[ \frac{1 - A}{2} y(k_1) - c(k_1) \right]
\]

\[
= \frac{1}{2} [\varphi(A) y(k_1) - 2c(k_1)],
\]

uniformly in \( k_1 \) and \( A \) as \( \sigma \) goes to zero. Under assumption
\[
\inf_{\theta \in (0,1]} \frac{2c(\theta)}{y(\theta)} < \varphi(-2 + \sqrt{9 - 24c'/y'}) \, \text{clearly} \, \varphi(A) y(k_1) - 2c(k_1) = 0 \text{ has a solution} \, k'_1 \in (\theta, 1) \, \text{by the continuity of} \, y, c. \text{Therefore, for a sufficiently small} \, \sigma > 0, \text{there exist} \, k_1, k_2 \,(0 < k_1 < k_2 < 1) \, \text{such that} \, u(k_2, I(k_1, k_2)) = u(k_1, I(k_1, k_2)) = 0.
\]

Finally it is easily checked that by the linearity of \( y, c \) that
\[
\lim_{x \to k_1^+} \frac{d}{dx} u(x, I(k_1, k_2)) > 0, \quad \lim_{x \to k_2^-} \frac{d}{dx} u(x, I(k_1, k_2)) < 0 \quad \text{and} \quad \frac{d^2}{dx^2} u(x, I(k_1, k_2)) < 0, k_1 < x < k_2.
\]

This shows \( u(x, I(k_1, k_2)) > 0 \) for \( k_1 < x < k \) and the proof is completed. \( \square \)

**A.2 Proofs for Section 4**

The optimal strategy for type B in Case 1 is obtained in the same way as in section 2. We denote by \( u(x, s, \alpha, \sigma) \) the expected payoff to production of an individual of type B observing signal \( x \) when the strategy for type B is \( s \) and the proportion of type-S individuals who choose to produce is \( \alpha \) \( (\alpha = 0, 1) \).

**Lemma 3**: When \( \alpha = 0 \) \( (\alpha = 1) \), for any \( \delta > 0 \) and \( \rho \in (\max_{0 < \delta < 1} \frac{2c'(\theta)}{y(\theta)}, 1] \) (resp.
any $\delta > 0$, there exists a $\sigma_1 > 0$ such that $u(x, I_{(s,1]}, \alpha, \sigma)$ is strictly increasing in $x$ on $(\delta, 1 - \delta)$ for all $(\sigma, \rho) \in (0, \sigma_1] \times [\rho, 1] \text{ (resp. } (\sigma, \rho) \in (0, \sigma_1] \times [0,1]).$

**Proof of lemma 3.**

$$u(x, I_{(s,1]}, \alpha, \sigma) = \frac{1}{\int_{-\sigma}^{x+\sigma} f(x-\theta) g(\theta) d\theta} \int_{-\sigma}^{x+\sigma} \left[ \rho (1 - F(\frac{x-\theta}{\sigma})) + (1 - \rho) x \alpha \right] y(\theta) - c(\theta) \right] f(\frac{x-\theta}{\sigma}) g(\theta) d\theta$$

$$= \frac{1}{m(x)} \int_0^1 dz H(x,z,\alpha) f(z) g(x-\sigma z),$$

where $m(x) = \int_0^1 dz f(z) g(x-\sigma z),$ and $H(x,z,\alpha) = \{ \rho (1 - F(z)) + (1 - \rho) x \alpha \} y(x - \sigma z) - c(x - \sigma z).$

Then

$$\frac{d}{dx} u(x, I_{(s,1]}, \alpha, \sigma) = \frac{1}{m^2} \left[ \int_0^1 dz \left( \frac{\partial H}{\partial x} g(x-\sigma z) + H g'(x-\sigma z) f(z) \right) might.$$  

$$- \left[ \int_0^1 dz H g(x-\sigma z) f(z) \right] \left[ \int_0^1 dz g'(x-\sigma z) f(z) \right] \right]$$

Since $H, \frac{\partial H}{\partial x}, g, g', f$ are uniformly continuous in $x, z, \sigma$ on $[\theta - \sigma, 1 - \delta] \times [-1,1] \times [0,\sigma_0]$, $\frac{d}{dx} u(x, I_{(s,1]}, \alpha, \sigma)$ converges to

$$\frac{d}{dx} u(x, I_{(s,1]}, \alpha, 0) = \int_0^1 dz \left[ \rho (1 - F(z)) + (1 - \rho) x \alpha \right] y'(x) - c'(x) f(z)$$

$$= \left\{ \frac{1}{2} \rho + (1 - \rho) \alpha \right\} y'(x) - c'(x)$$

uniformly for all $x \in [\delta, 1 - \delta]$ as $\sigma \to +0$.

Hence, the conclusion follows from
\[
\frac{d}{dx} u(x, l_{1, r_{1}}(x), \alpha, 0) > \left( \min_{0 \leq s \leq 1} y'(x) \right) \left( \frac{1}{2} \rho_1 - \max_{0 \leq s \leq 1} \frac{c'(x)}{y'(x)} \right) > 0
\]
for \( x \in [\delta, 1 - \delta] \) and \( \rho \in [\rho_0, 1] \) when \( \alpha = 0 \), and
\[
\frac{d}{dx} u(x, l_{1, r_{1}}(x), \alpha, 0) \geq \left( \min_{0 \leq s \leq 1} y'(x) \right) \left( \frac{1}{2} - \max_{0 \leq s \leq 1} \frac{c'(x)}{y'(x)} \right) > 0
\]
for \( x \in [\delta, 1 - \delta] \) and \( \rho \in [0, 1] \) when \( \alpha = 1 \).

**Lemma 4:** When \( \alpha = 0 \) (\( \alpha = 1 \)), there exist \( \delta, \sigma_2, \rho_1 \) with \( \delta > \sigma_2 > 0 \) \( \text{and} \ 1 > \rho_1 \geq \frac{2c(1)}{y(1)(1 - F(0))} \) (resp. \( \delta, \sigma_2 \) with \( \delta > \sigma_2 > 0 \)) such that \( u(x, l_{1, r_{1}}(x), \alpha, \sigma) \) is positive uniformly in \( x, \sigma_1, \rho \) on \([1 - \delta, 1) \times (0, \sigma_2) \times [\rho_1, 1] \) (resp. \([1 - \delta, 1) \times (0, \sigma_2) \times [0, 1] \)).

**Proof of Lemma 4.** We prove only the case of \( \alpha = 0 \). It is shown in the same way for the case of \( \alpha = 1 \).

By the uniform continuity of \( g \), we choose \( \sigma_2 > 0 \) such that \( |x_1 - x_2| \leq \sigma_2 \) means \( |g(x_1) - g(x_2)| \leq \varepsilon < \min_{0 \leq s \leq 1} g(x) \). Further, we choose \( \delta > \sigma_2 \) such that \( y(1 - \delta - \sigma_2) > \frac{2c(1)}{\rho_1 \gamma 1 - F(0)} \).

Then, when \( \alpha = 0 \), for any \((x, \sigma)\) such that either \( 1 - \delta \leq x \leq 1 - \sigma, 0 < \sigma \leq \sigma_2 \) or \( x < 1 - \delta, x_0 < \sigma \leq \sigma_2 \),

\[
u(x, l_{1, r_{1}}(x), \alpha, \sigma) > \int_0^1 dz \left[ \rho (1 - F(z)) y(1 - \delta - \sigma) - c(l) \right] (g(x) - \varepsilon) f(z) \geq \min_{0 \leq s \leq 1} \frac{g(x) - \varepsilon}{g(x) + \varepsilon} \left[ \frac{1}{2} \rho_1 y(1 - \delta - \sigma) - c(l) \right] > 0.
\]

As for any \((x, \sigma)\) such that \( 0 < 1 - x \leq \sigma \leq \sigma_2 \), noting

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\[ \int_{-\infty}^{\infty} F(z) f(z) dz = \frac{1}{2} (1 - F(\frac{x-1}{\sigma})) \] we have
\[ \mu(x,I_{(x,1)},\alpha,\sigma) \]
\[ \geq \min_{0 \leq i \leq 1} \left\{ \frac{g(x) - \varepsilon}{g(x) + \varepsilon} \right\} \left[ \frac{1}{2} \rho(1 - F(0)) \gamma(1 - \delta - \sigma_2) - \varepsilon(1) \right] > 0. \]

Proposition 4 is straightforward from the following lemma.

**Lemma 5:** When either \( \alpha = 0 \), \( \rho \in (\rho,1] \) or \( \alpha = 1 \), \( \rho \in (0,1] \), \( \mu(x,I_{(x,1)},\alpha,\sigma) \) has a unique zero point \( k(\alpha,\sigma,\rho) \) in \((0,1)\), and \( s = I_{(k(\alpha,\sigma,\rho),1)} \) is an optimal strategy for individuals of type B. In particular, that \( \alpha = 1 \) and \( 0 < \rho < \rho^* \) is necessary and sufficient in order that \( s = I_{(k(\alpha,\sigma,\rho),1)} \) is a unique optimal strategy for type B.

**Proof of lemma 5.** By lemma 3 and lemma 4, when either \( \alpha = 0 \), \( \rho \in (\rho,1] \) or \( \alpha = 1 \), \( \rho \in (0,1] \), there exist \( \delta, \sigma_j > 0 \) such that \( \delta < 0 - \sigma_3 \), \( \mu(x,I_{(x,1)},\alpha,\sigma) > 0 \) for \( (x,\sigma) \in [1-\delta,1) \times (0,\sigma_j) \) and \( \mu(x,I_{(x,1)},\alpha,\sigma) \) is strictly increasing on \((\delta,1-\delta)\). This shows \( \mu(x,I_{(x,1)},\alpha,\sigma) \) has a unique zero point \( k(\alpha,\sigma,\rho) \) in \((0,1)\) since \( \mu(x,I_{(x,1)},\alpha,\sigma) \) is continuous on \((0,1)\) and negative on \((0,\delta)\).

The optimality of \( s = I_{(k,1)} \) is proved by the exact same argument as lemma 2, since \( \lim_{x \to 1^-} \mu(x,I_{(k,1)},\alpha,\sigma) > 0 \) by lemma 4.
For the second part, it is easily shown from

\[
\begin{align*}
&u(1,s,1,\sigma) = \int_{0}^{\frac{\theta}{\sigma}} (1 - F(\frac{1 - \theta}{\sigma}))g(\theta)d\theta \\
&\quad - \int_{s}^{1} (1 - F(\frac{1 - \theta}{\sigma}))g(\theta)d\theta \\
&\quad + \left[ (\rho(\frac{\theta}{\sigma}) (\frac{1 - \theta}{\sigma})d\theta + (1 - F(\frac{1 - \theta}{\sigma})s(1)) + (1 - \rho) \right] y(\theta) - c(\theta) \\
&\quad \frac{1}{1}
\end{align*}
\]

that \( \alpha = 1 \) and \( 0 < \rho < \bar{\rho} \) is necessary and sufficient in order that \( u(1,s,1,\sigma) \) is uniformly positive for all \( s \) and all \( (\sigma, \rho) \in (0, \sigma_x) \times (0, \bar{\rho}) \) for some \( \sigma_x > 0 \). This implies that \( \alpha = 1 \) and \( 0 < \rho < \bar{\rho} \) is necessary and sufficient in order that \( s = I_{(\delta,1,\sigma),\rho},1 \) is a unique optimal strategy.

\[\square\]

**Proof of proposition 5.** We first note that \( \alpha \) should be taken as the following continuous function of \( x \) in Case 2:

\[
\alpha(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \theta_r - 2\sigma \\
\int_{0}^{\theta_r - 2\sigma} dz g(z-x) & \text{if } \theta_r - 2\sigma < x < \theta_r + 2\sigma \\
\int_{0}^{\theta_r + 2\sigma} dz g(z-x) & \text{if } \theta_r + 2\sigma \leq x \leq 1 \\
\end{cases}
\]

It then becomes obvious that \( u(x,l_{(\delta,1,\sigma),\rho}),\alpha(x),\sigma) \) coincides \( u(x,l_{(\delta,1,\sigma),0},\sigma) \) ( \( u(x,l_{(\delta,1,\rho)},\sigma) \) ) on \( (0, \theta_r - 2\sigma) \) (resp. \( (0, \theta_r + 2\sigma) \) ), and

\[
u(x,l_{(\delta,1,\sigma),0},\sigma) < u(x,l_{(\delta,1,\sigma),\alpha(x),\sigma}) < u(x,l_{(\delta,1,\rho),\sigma}) \text{ on } (0, \theta_r - 2\sigma) \text{ for all } \sigma > 0.
\]

Further, \( u(x,l_{(\delta,1,\sigma),\alpha(x),\sigma}) \) is strictly increasing in \( x \) on \( (\delta,1-\delta) \setminus [\theta_r - 2\sigma, \theta_r + 2\sigma] \) by lemma 3. Therefore, the first part of the proposition follows from lemma 5 if we show that \( u(x,l_{(\delta,1,\sigma),\alpha(x),\sigma}) \) never has
any zero point in \([\theta_r - 2\sigma, \theta_r + 2\sigma]\) for any sufficiently small \(\sigma > 0\). But this easily follows from that \(u(x, l_{(x,1)}, l, \sigma)\) (resp. \(u(x, l_{(x,1)}, 0, \sigma)\) ) converges to 
\[
(1 - \frac{1}{2} \rho) y(x) - c(x) \quad \text{(resp. } \frac{1}{2} \rho y(x) - c(x) \text{)}
\]
uniformly in \(x\).

The second part also follows from lemma 5 since \(u(x, l_{(x,1)}, \alpha(x), \sigma)\) coincides with \(u(x, l_{(x,1)}, l, \sigma)\) in the neighborhood of \(1\). \(\Box\)
References


