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Based on Sequential Monte Carlo Methods**

by

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# **Time-varying Analysis of Dynamic Stochastic General Equilibrium Models Based on Sequential Monte Carlo Methods**

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## **Abstract**

This paper proposes a new method to estimate parameters, natural rates, and business cycles of dynamic stochastic general equilibrium models simultaneously and consistently. It is based on the Monte Carlo particle filter and a self-organizing state space model. In our method, we estimate the parameters and the natural rates using the time-varying-parameter approach, which is often used to infer invariant parameters practically. In most previous papers on DSGE models, structural parameters of them are assumed to be “deep (invariant).” However, our method analyzes how stable structural parameters are. Adopting the TVP approach creates the great advantage that the structural changes of parameters are detected naturally. Moreover, we estimate time-varying natural rates of macroeconomic data: real output and an equilibrium real interest rate. In empirical analysis, we estimate a new Keynesian DSGE model using the US data. The analysis shows that the average of the growth of natural output is 3.02 and the average of inflation target is 2.46 from 1985 to 2007. From the late Volcker era to the early Greenspan era, the reaction coefficient to inflation in the Taylor rule is increasing, and from the mid Greenspan era to the early Bernanke era it is stable around 4.0. These results indicate that the behavior of the Fed had changed to realize the stability of inflation from the late Volcker era to the early Greenspan era. An equilibrium real rate is negative from the early 2000s to the mid-2000s because the Fed cuts policy rates to prevent deflation.

JEL Classification Number: C11, C15, E00, E52, and D50

Key words : Dynamic Stochastic General Equilibrium Model, Bayesian Statistics, Monte Carlo particle filter, Self-organizing state space model, US economy

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# 1 Introduction

In recent years, new Keynesian, dynamic stochastic general equilibrium models of monetary analysis have been rapidly developing. The early works of Kimball (1995), Roberts (1995), and Yun (1996) beget the subsequent many papers (see McCallum and Nelson (1999), Clarida et al. (1999), Gali (2002), and related studies which are referred therein)<sup>2</sup>. “Middle-size” new Keynesian models are developed by Christiano et al. (2005) and Smets and Wouters (2003), and their models are often adopted by practitioners in the government and the central bank. The fit performance of their models is discussed by Fout (2005), Trabandt (2006), and Del Negro et al. (2007).

Bayesian statistics are now becoming a standard tool to estimate these DSGE models. Smets and Wouters (2003), Onatski and Williams (2004), Levin et al. (2005), Del Negro et al. (2007), Smets and Wouters (2007), Hirose and Naganuma (2007), and many studies estimate parameters of new Keynesian DSGE models using Markov Chain Monte Carlo methods (MCMC). Fernandez-Villaverde and Rubio-Ramirez (2005) and Fernandez-Villaverde and Rubio-Ramirez (2007a) have shown that the Monte Carlo particle filter (MCPF) and maximizing likelihood can be successfully applied to estimate DSGE models<sup>3</sup>. An and Schorfheide (2007) is an excellent survey on this area, and see references cited therein<sup>4</sup>.

This paper proposes a new method to estimate parameters, natural rates, and business cycles of dynamic general equilibrium models simultaneously and consistently. It is based on the Monte Carlo particle filter, proposed by Kitagawa (1996) and Gordon et al. (1993), and a self-organizing state space model, proposed by Kitagawa (1998)<sup>5</sup>. In our method, we estimate the parameters and the natural rates using the time-varying-parameter approach, which is often used to infer invariant parameters practically. Adopting the TVP approach creates the great advantage that the structural changes of parameters are detected naturally. In most previous papers on DSGE models, structural parameters of them are assumed to be “deep (invariant).” However, our method analyzes how stable structural parameters are. Moreover, we estimate time-varying natural rates of macroeconomic data: natural output, a inflation rate, and a real interest rate<sup>6</sup>. To estimate natural rates of macroeconomic data, the Hodrick and Prescott (1997) filter, the Baxter and King (1999) filter, the Christiano and Fitzgerald (2003) filter, and other filtering algorithms are also often used in practice. Our method, however, is an alternative to these filters, and it is “structural” estimation of time-varying natural rates. Again, we would like to stress that our method simultaneously and consistently estimates natural rates, time-varying parameters, and business cycles.

In empirical analysis, we estimate a new Keynesian DSGE model using the US data from 1981:Q1 to 2007:Q4. The analysis shows that the average of the growth of natural output is 3.02 and the average of inflation target is 2.46 from 1985 to 2007. From the late Volcker era to the early Greenspan era, the reaction coefficient to inflation in the Taylor rule is increasing, and from the mid Greenspan era to the early Bernanke era it is stable around 4.0%. It also indicates that the behavior of the Fed had changed to realize mild inflation from the late Volcker era to the

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<sup>2</sup>See also Walsh (2003), Woodford (2003), Kato (2006), Gali (2008), and related studies cited therein.

<sup>3</sup>Amisano and Tristani (2007) estimates a small DSGE model on euro area data, using the conditional particle filter to compute the model likelihood.

<sup>4</sup>Canova (2007) and Dejong and Dave (2007) are comprehensive introductions to Bayesian macroeconometrics. Using MCMC, Iiboshi et al. (2005), Sugo and Ueda (2008), and Ichiue et al. (2008) estimate DSGE models for Japan in the “pre-zero-interest-rate” period (1970[1981]-1995).

<sup>5</sup>Introductions to Monte Carlo particle filters are Gordon et al. (1993), and Doucet et al., eds (2001), Ristic et al. (2004). Yano (2008b) and Yano and Yoshino (2007) propose time-varying structural vector autoregressions based on the Monte Carlo particle filter and a self-organizing state space model. Time-varying structural vector autoregressions based on Markov chain Monte Carlo methods are proposed by Primiceri (2005) and Canova and Gambetti (2006).

<sup>6</sup>Smets and Wouters (2007) estimate invariant trends of macroeconomic data. Edge et al. (2008) propose new method to estimate natural rates, and Adjemian and Juillard (2009) adopt the diffuse Kalman filter, proposed by Koopman and Durbin (2003). Justiniano and Primiceri (2008b) analyze potential and natural output using DSGE models. Yano (2009) extends our method to estimate DSGE models in a liquidity trap.

early Greenspan era. An equilibrium real rate is negative from the early 2000s to the mid-2000s because the Fed cuts policy rates to prevent deflation. In our estimation, most structural variables are stable. However, The inverse of the elasticity of labor,  $\sigma_L$ , is varying from about 2.0 to about 3.5, and it is consist that the estimates of  $\sigma_L$ , which are inferred by Smets and Wouters (2003), Levin et al. (2005), Del Negro et al. (2007), Smets and Wouters (2007), and related studies, have relatively large standard deviations.

Adopting the time-varying-parameter approach does not suggests that structural parameters are “time-varying.” In most previous papers on DSGE models, structural parameters of them are assumed to be “deep (invariant).” However, our method analyzes how stable structural parameters are. The TVP approach is practically often used in state space modeling to estimate parameters, for example, Kitagawa (1998) and Liu and West (2001). Even if we assume the random walk priors, which are described in appendix C, it does not indicate that the deep parameters of DSGE models are “time-varying.” Our framework is just a practical one to estimate deep parameters. Adopting the TVP approach creates the great advantage that the structural changes of parameters are detected naturally. Thus, our method is suitable to analyze how stable structural parameters are.

Our paper is closely related with Fernandez-Villaverde and Rubio-Ramirez (2007b) <sup>7</sup>. However, there exist several large differences between our paper and theirs. The first point is that they focus on the stabilities of parameters of the Taylor rule and several parameters, for example, Calvo pricing parameters. In contrast our main concern is that natural rates, deep parameters, business cycles are simultaneously and consistently estimated using the TVP approach. The second point is that they use maximizing the likelihood of MCPF to estimate parameters, while, we adopt a self-organizing state space model for estimation. Yano (2008a) reports that the variances of the estimates of a self-organizing state space model are smaller than the ones of the maximizing-likelihood approach. The third point is that we estimate a time-varying natural rate of real output, a time-varying inflation target, and a time-varying equilibrium real interest rate.

Canova and Sala (2009) point out that structural parameters, which is based on maximum likelihood estimation, of DSGE models is weakly identified. The main cause of the problem is that the curvature of the likelihood of a DSGE model and the population objective function may be small in certain regions of the parameter space. In our method, the search space of most time-varying parameters is possible to be strictly limited in a “reasonable” region because the MCP filter is a simulation-based algorithm. For example, the search space of the time-varying reaction parameter to inflation in the Taylor rule is limited from 1.0 to 3.0 because the range is consistent with the Taylor principle and the results of previous studies, for example, Del Negro et al. (2007) and Smets and Wouters (2007). Such kind of restrictions based on previous studies are “reasonable” and help to reduce the problem.

This paper is structured as follows. In section 2, we describe a new Keynesian DSGE model. In section 3, we explain our method based on the Monte Carlo particle filter and a self-organizing state space model. In section 4, we show the results of our empirical analysis. In section 5, we describe conclusions and discussions.

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<sup>7</sup>Justiniano and Primiceri (2008a) estimate DSGE models allowing for time variation in the volatility of the structural innovations using MCMC. Bjornland et al. (2008) estimate the time-varying natural rate of interest and output and the implied medium-term inflation target for the US economy based on DSGE models using MCMC and the Kalman filter. Juillard (2008) estimates trends in Cash-in-advance DSGE models using the diffuse Kalman filter, proposed by Koopman and Durbin (2003). Hatano (2004) and Hatano (2009) estimate structural parameters of a overlapping generations model using the Kalman filter. Ferroni (2009) proposes a detrending method based on Bayesian statistics.

## 2 The Model

### 2.1 Households

In the economy, there is a continuum of households indexed by  $j \in (0, 1)$ . The households consume and provide labor. The utility of the household  $j$  is given by

$$E_0^j \sum_{t=0}^{\infty} \beta^t \left[ \log(C_{j,t} - hC_{t-1}) + \Upsilon \left( \frac{M_{j,t}}{P_t} \right) - \frac{\Psi_L}{1 + \sigma_L} L_{j,t}^{1+\sigma_L} \right], \quad (1)$$

where  $E_0^j$  is the expectation operator, conditional on household  $j$ 's information at time 0,  $C_{j,t}$  is household  $j$ 's consumption,  $C_{t-1}$  is past aggregate consumption,  $M_{j,t}/P_t$  is the household  $j$ 's real money balances,  $L_{j,t}$  is household  $j$ 's labor hours,  $t$  is a time index, and  $h$ ,  $\chi$ ,  $\Psi_L$ , and  $\eta$  are constants. The constraint condition of the household  $j$  is given by

$$C_{j,t} + I_{j,t} + \frac{M_{j,t}}{P_t} + \frac{B_{j,t}}{P_t} \leq W_t L_{j,t} + \frac{M_{j,t-1}}{P_t} + r_t^K K_{j,t} + (1 + i_{t-1}) \frac{B_{j,t-1}}{P_t} + \Pi_{j,t}, \quad (2)$$

where  $I_{j,t}$  is investment by household  $j$ ,  $B_{j,t}$  is household  $j$ 's domestic bonds,  $W_t$  is the average real wage,  $i_t$  is the short-term nominal interest rate,  $K_{j,t}$  is household  $j$ 's capital,  $r_t^K$  is the rental rate of  $K_{j,t}$ , and  $\Pi_{j,t}$  is the profit of the firm  $j$ . In addition to Eq. (2), we assume the households are subject to the no-Ponzi condition.

$$\lim_{T \rightarrow \infty} E_0 \left[ \left( \prod_{t=0}^T \frac{1}{1 + i_t} \right) B_{j,T} \right] = 0. \quad (3)$$

### 2.2 Capital Accumulation and Adjustment Cost

The time evolution of Capital,  $K_{j,t}$  is given by

$$K_{j,t} = (1 - \delta)K_{j,t-1} + \left[ 1 - s \left( \frac{I_{j,t}}{I_{j,t-1}} \right) \right] I_{j,t}, \quad (4)$$

where  $\delta$  is the depreciation cost of capital,  $K_{j,t}$ , and  $s(\cdot)$  is a adjustment cost function. We restrict the function  $s(\cdot)$  to satisfy the following properties:  $s(1) = s'(1) = 0$  and  $s''(1) = 1/\nu > 0$ .

### 2.3 Final Good Sector

In the final good sector, a single final good is produced by a perfectly competitive, representative firm. The final good is produced using a continuum of intermediate good,  $Y_{j,t}$ , indexed by  $j \in (0, 1)$ . The final good,  $Y_t$ , is produced using the aggregate technology.

$$Y_t = \left[ \int_0^1 (Y_{j,t})^{\frac{1}{1+\lambda_p}} dj \right]^{1+\lambda_p}, \quad (5)$$

where  $Y_{j,t}$  is the quantity of intermediate good  $j$ ,  $\lambda_p$  is a constant. The demand curve for  $Y_{j,t}$  is given by

$$Y_{j,t} = \left( \frac{P_{j,t}}{P_t} \right)^{-\frac{1+\lambda_p}{\lambda_p}} Y_t, \quad (6)$$

where  $P_{j,t}$  is the price of intermediate good  $j$  and  $P_t$  is the aggregate price of the final good. The aggregate price is given by

$$P_t = \left[ \int_0^1 (P_{j,t})^{-\frac{1}{\lambda_p}} dj \right]^{-\lambda_p}. \quad (7)$$

## 2.4 Intermediate Goods Firms

In the intermediate goods sector, monopolistic competitive domestic firms produce intermediate goods which is indexed by  $j \in (0, 1)$ . The firm  $j$ 's production function is given by

$$Y_{j,t} = Z_t K_{j,t}^\alpha L_{j,t}^{1-\alpha}, \quad (8)$$

The aggregate technology level,  $Z_t$ , is given by

$$\log Z_t = (1 - \xi_Z) \log \bar{Z} + \xi_Z \log Z_{t-1} + \epsilon_{Z,t}, \quad (9)$$

where  $\epsilon_{Z,t} \sim N(0, \sigma_{Z,t}^2)$  and  $\bar{Z}$  and  $\xi_Z$  are constants. Solving the cost minimization of the firm  $j$ , the first order condition becomes

$$\frac{W_t}{r_t^K} = \frac{1 - \alpha}{\alpha} \frac{K_{j,t}}{L_{j,t}} \quad (10)$$

The firms  $j$ 's real marginal cost is given by

$$MC_t = \frac{1}{Z_t} \left( \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)} W_t^{1-\alpha} (r_t^K)^\alpha \right) \quad (11)$$

In the sticky prices model, proposed by Calvo (1983), a fraction  $1 - \xi_p$  of all firms re-optimize their nominal prices while the remaining  $\xi_p$  fraction of all firms do not re-optimize their nominal prices. Following Christiano et al. (2005), firms that cannot re-optimize their price index to lagged inflation are as follows.

$$P_{j,t} = \pi_{t-1} P_{j,t-1}, \quad (12)$$

where  $\pi_t = P_t/P_{t-1}$ . We call this price setting ‘‘lagged inflation indexation.’’ The firm  $j$  chooses  $P_{j,t}$  to maximize

$$E_t \sum_{l=0}^{\infty} (\beta \xi_p)^l \left[ \frac{P_{j,t}}{P_{t+l}} X_{tl} - MC_{t+l} \right] Y_{j,t+l}, \quad (13)$$

subject to  $Y_{j,t} = \left( \frac{P_{j,t}}{P_t} \right)^{-\frac{1+\lambda_p}{\lambda_p}} Y_t,$

where  $X_{tl}$  is

$$X_{tl} = \begin{cases} \pi_t \times \pi_{t+1} \times \cdots \times \pi_{t+l-1} & \text{for } l \geq 1 \\ 0 & \text{for } l = 0. \end{cases} \quad (14)$$

The aggregate price index of sticky prices and inflation indexation is obtained by

$$P_t = \left[ (1 - \xi_p) (\tilde{P}_t)^{\frac{1}{1-\lambda_p}} + \xi_p (\pi_{t-1} P_{t-1})^{\frac{1}{1-\lambda_p}} \right]^{1-\lambda_p}. \quad (15)$$

## 2.5 Monetary Policy

The monetary authority is assumed to determine the nominal interest rate according to the Taylor rule, proposed by Taylor (1993).

$$i_t = (i_{t-1})^{\rho_i} (Y_t^{\phi_Y} \pi_t^{\phi_\pi})^{1-\rho_i} \quad (16)$$

where  $\phi_Y$  and  $\phi_\pi$  are constants and  $\epsilon_{i,t} \sim N(0, \sigma_{i,t}^2)$ .

## 2.6 Market Clearing

In the final market equilibrium, the final good production is equivalent to the households' demand for consumption, investment, and the expenditure of the government.

$$Y_t = C_t + I_t + G_t, \quad (17)$$

where  $Y_t = \int_0^1 Y_{j,t} dj$ ,  $C_t = \int_0^1 C_{j,t} dj$ ,  $I_t = \int_0^1 I_{j,t} dj$ , and  $G_t$  is a government expenditure.

## 2.7 Linearized Model

We linearize the model described above around the non-stochastic steady state. The linearized model consists of the new IS curve (NISC), the new Keynesian Phillips curve (NKPC), the Taylor rule (TR), and several equations. NISC is obtained as follows <sup>8</sup>.

$$\hat{C}_t = \frac{h}{1+h} \hat{C}_{t-1} + \frac{1}{1+h} E_t \hat{C}_{t+1} - \frac{1}{1+h} E_t [\hat{i}_t - \hat{\pi}_{t+1}] + \epsilon_{C,t}. \quad (18)$$

NKPC is obtained as follows.

$$\hat{\pi}_t = \frac{1}{1+\beta} \hat{\pi}_{t-1} + \frac{\beta}{1+\beta} E_t \hat{\pi}_{t+1} + \frac{(1-\xi_p)(1-\beta\xi_p)}{\xi_p(1+\beta)} [(1-\alpha)\hat{W}_t + \alpha\hat{r}_t^K - \hat{Z}_t] + \epsilon_{\pi,t}. \quad (19)$$

The other equations are

$$\hat{W}_t = \sigma_C (\hat{Y}_t - h\hat{Y}_{t-1}) + \sigma_L \hat{L}_t, \quad (20)$$

$$\hat{L}_t = -\hat{W}_t + \hat{r}_t^K + \hat{K}_t, \quad (21)$$

$$\hat{I}_t = \frac{1}{1+\beta} \hat{I}_{t-1} + \frac{\beta}{1+\beta} \hat{I}_{t+1} + \frac{\nu}{1+\beta} \hat{Q}_t + \epsilon_{I,t}, \quad (22)$$

$$\hat{Q}_t = -E_t [\hat{i}_t - \hat{\pi}_{t+1}] + \frac{1-\delta}{1-\delta+\bar{r}^K} E_t \hat{Q}_{t+1} + \frac{\bar{r}^K}{1-\delta+\bar{r}^K} E_t \hat{r}_{t+1}^K + \epsilon_{Q,t}, \quad (23)$$

$$\hat{K}_t = (1-\delta)\hat{K}_{t-1} + \delta\hat{I}_{t-1}, \quad (24)$$

$$\hat{Y}_t = \Psi_C \hat{C}_t + \Psi_I \hat{I}_t + \Psi_G \hat{G}_t, \quad (25)$$

$$\hat{Y}_t = \hat{Z}_t + \alpha\hat{K}_t + (1-\alpha)\hat{L}_t, \quad (26)$$

$$\hat{G}_t = \rho_G \hat{G}_{t-1} + \epsilon_{G,t}, \quad (27)$$

<sup>8</sup>In this paper, a hat over a variable indicates the percentage deviation from its steady state value.

and

$$\hat{Z}_t = \xi_Z \hat{Z}_{t-1} + \epsilon_{Z,t}, \quad (28)$$

where  $\epsilon_{C,t} \sim N(0, \sigma_{C,t}^2)$ ,  $\epsilon_{\pi,t} \sim N(0, \sigma_{\pi,t}^2)$ ,  $\epsilon_{I,t} \sim N(0, \sigma_{I,t}^2)$ ,  $\epsilon_{Q,t} \sim N(0, \sigma_{Q,t}^2)$ ,  $\epsilon_{G,t} \sim N(0, \sigma_{G,t}^2)$ , and  $\epsilon_{Z,t} \sim N(0, \sigma_{Z,t}^2)$ . Note that  $\Psi_C = \bar{C}/\bar{Y}$ ,  $\Psi_I = \bar{I}/\bar{Y}$ , and  $\Psi_G = \bar{G}/\bar{Y}$ . Thus, the linearized TR is given by

$$\hat{i}_t = \rho_i \hat{i}_{t-1} + (1 - \rho_i)(\phi_Y \hat{Y}_t + \phi_\pi \hat{\pi}_t) + \epsilon_{i,t}, \quad (29)$$

where  $r^s$  is an equilibrium real rate and  $\pi^s$  is the target rate of inflation.

## 2.8 State Space Model

Structural linear rational expectations models are given by

$$\mathbf{\Gamma}_0 \mathbf{x}_t = \mathbf{\Gamma}_1 \mathbf{x}_{t-1} + \mathbf{\Psi} \mathbf{z}_t + \mathbf{\Pi} \boldsymbol{\eta}_t + \mathbf{C}, \quad (30)$$

where  $\mathbf{x}_t = [E_t \hat{C}_{t+1}, E_t \hat{\pi}_{t+1}, E_t \hat{r}_{t+1}^K, E_t \hat{I}_{t+1}, E_t \hat{Q}_{t+1}, \hat{Y}_t, \hat{C}_t, \hat{i}_t, \hat{\pi}_t, \hat{W}_t, \hat{r}_t^K, \hat{Z}_t, \hat{L}_t, \hat{K}_t, \hat{I}_t, \hat{Q}_t, \hat{G}_t]^t$ ,  $\mathbf{z}_t = (\epsilon_{C,t}, \epsilon_{\pi,t}, \epsilon_{I,t}, \epsilon_{Q,t}, \epsilon_{i,t}, \epsilon_{G,t}, \epsilon_{z,t})^T \sim N(\mathbf{0}, \mathbf{\Sigma}_t)$  with  $\mathbf{\Sigma}_t = \text{diag}((\sigma_{C,t})^2, (\sigma_{\pi,t})^2, (\sigma_{I,t})^2, (\sigma_{Q,t})^2, (\sigma_{i,t})^2, (\sigma_{G,t})^2, (\sigma_{z,t})^2)$ ,  $\mathbf{z}'_t = (\epsilon_{C,t}, \epsilon_{\pi,t}, \epsilon_{I,t}, \epsilon_{Q,t}, \epsilon_{G,t}, \epsilon_{z,t})^T \sim N(\mathbf{0}, \mathbf{\Sigma}_t)$  with  $\mathbf{\Sigma}_t = \text{diag}((\sigma_{C,t})^2, (\sigma_{\pi,t})^2, (\sigma_{I,t})^2, (\sigma_{Q,t})^2, (\sigma_{G,t})^2, (\sigma_{z,t})^2)$ ,  $\mathbf{\Pi} = \mathbf{0}$ , and  $\mathbf{C} = \mathbf{0}$ <sup>9</sup>. Sims (2002) proposes the solution of linear rational expectations models using QZ decomposition<sup>10</sup>. Following Sims (2002), reduced linear rational expectations models are obtained by

$$\mathbf{x}_t = \mathbf{\Theta}_1 \mathbf{x}_{t-1} + \boldsymbol{\epsilon}_{1,t}, \quad (31)$$

where  $\boldsymbol{\epsilon}_{1,t} = \mathbf{\Theta}_0 \mathbf{z}_t$  and  $\boldsymbol{\epsilon}'_{1,t} = \mathbf{\Theta}'_0 \mathbf{z}'_t$ . The symbols,  $\mathbf{\Theta}_1$ ,  $\mathbf{\Theta}_0$ ,  $\mathbf{\Theta}'_1$ , and  $\mathbf{\Theta}'_0$  are described in Sims (2002).

The measurement equation of the model is

$$\mathbf{Y}_t = \mathbf{Y}^s + \mathbf{H} \mathbf{x}_t + \mathbf{v}_t, \quad (32)$$

where  $\mathbf{Y}_t = [YGR_t, CGR_t, IGR_t, WGR_t, INFL_t, LGR_t, INT_t]^t$ ,  $\mathbf{Y}^s = [Y^s, Y^s, Y^s + I^s, Y^s, \pi^s, L^s, r^s + \pi^s]^t$ , and  $\mathbf{v}_t = (\epsilon_{Y,t}^v, \epsilon_{C,t}^v, \epsilon_{\pi,t}^v, \epsilon_{I,t}^v, \epsilon_{W,t}^v, \epsilon_{L,t}^v, \epsilon_{i,t}^v)^T \sim N(\mathbf{0}, \mathbf{\Sigma}_{v,t})$  with  $\mathbf{\Sigma}_{v,t} = \text{diag}((\sigma_{Y,t}^v)^2, (\sigma_{C,t}^v)^2, (\sigma_{\pi,t}^v)^2, (\sigma_{I,t}^v)^2, (\sigma_{W,t}^v)^2, (\sigma_{L,t}^v)^2, (\sigma_{i,t}^v)^2)$ <sup>11</sup>. The growth rate of real variables,  $YGR_t$ ,  $CGR_t$ ,  $IGR_t$ ,  $WGR_t$ , and  $LGR_t$ , are log differences of real GDP per capita, real consumption per capita, real investment per capita, real average wage, and average labor hours, respectively. The rate of inflation,  $INFL_t$ , is the a log difference of GDP deflator, and the nominal interest rate,  $INT_t$ , is the uncollateralized overnight call rate. Any observations are annualized. The symbols,  $Y^s$ ,  $\pi^s$ ,  $L^s$  and  $r^s$  are the growths of real output, the target rate of inflation, the trend of labor, and the trend of real interest rates, respectively. The term  $I^s$  denotes the investment-specific growth rate<sup>12</sup>.

<sup>9</sup>We set  $\mathbf{\Pi}$  to  $\mathbf{0}$  to rule out the indeterminacy and sunspot equilibrium, which are discussed in Sims (2002), Lubik and Schorfheide (2003), and Hirose (2007).

<sup>10</sup>In empirical analysis, we use Sims's gensys.R and related codes. See <http://sims.princeton.edu/yftp/gensys/>

<sup>11</sup>This equation is a modified version of the measurement equation of An and Schorfheide (2007).

<sup>12</sup>Greenwood et al. (1997) and Liu et al. (2008) analyze investment-specific technology changes in general equilibrium models, and related studies are cited therein.



In our method, we estimate the parameters of Eq. (18)- (32) using the TVP approach, which is explained in section 3. Thus, we define the vector of time-varying parameters as follows.

$$\tilde{\theta}_t = [h_t, \xi_{p,t}, \sigma_{L,t}, \nu_t, \xi_{Z,t}, \rho_{i,t}, \phi_{Y,t}, \phi_{\pi,t}, \rho_{G,t}, \sigma_{C,t}, \sigma_{\pi,t}, \sigma_{I,t}, \sigma_Q,t, \sigma_{i,t}, \sigma_{G,t}, \sigma_{Z,t}, \\ Y_t^s, I_t^s, \pi_t^s, L_t^s, r_t^s, \sigma_{Y,t}^v, \sigma_{C,t}^v, \sigma_{\pi,t}^v, \sigma_{I,t}^v, \sigma_{W,t}^v, \sigma_{L,t}^v, \sigma_{i,t}^v]. \quad (33)$$

Note that we calibrate seven parameters:  $\beta, \alpha, \delta, \bar{r}^K = 1/\beta - 1 + \delta, \Psi_{C,t}, \Psi_{I,t}$ , and  $\Psi_{G,t}$  (see section 3.1.) Reduced linear rational expectations models are also redefined by

$$\mathbf{x}_t = \Theta_{1,t} \mathbf{x}_{t-1} + \epsilon_{1,t}, \quad (34)$$

where  $\epsilon_{1,t} = \Theta_{0,t} \mathbf{z}_t$  and  $\epsilon'_{1,t} = \Theta'_{0,t} \mathbf{z}_t$ .

In previous papers on DSGE models, structural parameters of them are assumed to be “deep (invariant).” Our method, however, analyzes how stable structural parameters are. The time-varying-parameter approach is often used in state space modeling to estimate invariant parameters, for example, Kitagawa (1998) and Liu and West (2001). Even if we assume the random walk priors, which are described in appendix C, it does not indicate that the deep parameters are “time-varying.” Our framework is just a practical one to estimate deep parameters. Adopting our framework creates the great advantage that the structural changes of parameters are detected naturally. Thus, it is suitable to analyze how stable structural parameters are. The second advantage of our method is that we are able to estimate new Keynesian DSGE models in the liquidity trap (Krugman (1998)) because NNNSS, which is described in section 3, allows model switching.

## 2.9 Algorithm

In our method, we adopt not a smoothing algorithm but a filtering algorithm because the rational expectations hypothesis is consistent with the latter. If we use a smoothing algorithm to estimate time-varying parameters, the estimates of them include the information at times  $t + 1, t + 2, \dots$  which is not known at time  $t$ . Our method to estimate time-varying parameters of DSGE models is summarized as follows:

1. In time  $t$ , generate  $\mathbf{z}_t$  based on the results at time  $t - 1$ .
2. Using particles, the linear rational expectations system is solved to obtain the state transition equation Eq. (31).
3. If a particle implies indeterminacy (or non-existence of a stable rational expectations solution), then the weight of the particle,  $w_t^m$ , is set to zero.
4. If a unique stable solution exists, then the weight of the particle is calculated using Eq. (C7).
5. Resampling particles with sampling probabilities proportional to  $w_t^1, \dots, w_t^M$ .
6. Replace  $t$  with  $t + 1$ .
7. Go to 1.

### 3 Empirical Analysis

We use data from 1981:Q1 up to 2007:Q4<sup>13</sup>. The simulation settings used in empirical analysis are described in appendix B.

#### 3.1 Preliminary Setting

Following Smets and Wouters (2007), we calibrate four parameters:  $\beta = 0.99$ ,  $\alpha = 0.3$ ,  $\delta = 0.06$ ,  $\bar{r}^K = 1/\beta - 1 + \delta$ ,  $\Psi_{C,t} = 0.6$ ,  $\Psi_{I,t} = 0.2$ , and  $\Psi_{G,t} = 0.2$ . For preliminary setting for our method, we estimate our DSGE model using MCMC<sup>14</sup>. In Table 1, the estimates of MCMC are shown. For our method, we determine

Table 1: Preliminary Parameter Estimation Based on MCMC

	prior density	posterior mean	confidence interval	
h	beta	0.5619	0.4709	0.6687
$\xi_P$	beta	0.8092	0.7604	0.8568
$\sigma_L$	norm	0.3471	0.2451	0.4336
$\nu$	norm	0.1185	0.0618	0.1832
$\xi_Z$	beta	0.8839	0.8396	0.9328
$\rho_G$	beta	0.6585	0.5252	0.7818
$\rho_i$	norm	0.4370	0.3543	0.5313
$\phi_Y$	norm	0.0200	0.0024	0.0777
$\phi_\pi$	norm	1.0196	1.0044	1.0370

the prior distributions of time-varying parameters based on Table 1. The other simulation settings are described in appendix B.

In Table 2, we show the search space of TVPs to reduce the weakly identification problem, which is pointed out by Canova and Sala (2009). In our method, the space of some time-varying parameters can be strictly limited in a “reasonable” region because the MCP filter is a simulation-based algorithm. For example, the search space of the time-varying reaction parameter to inflation in the Taylor rule is limited from 1.0 to 3.0 because the range is consistent with the Taylor principle and the results of previous studies (Del Negro et al. (2007) and Smets and Wouters (2007)). If the particles are outside of the range, they are just discarded in simulation. Such kind of restrictions based on previous studies are “reasonable” and help to reduce the problem.

Figure 1 shows the annualized estimates of  $Y_t^s$ ,  $\pi_t^s$ , and  $r_t^s$ <sup>15</sup>. The black lines in all figures are means of particles, and the green and red lines are 95% confidence intervals, which are calculated using 100 bootstrap samples of particles. To estimate natural rates of macroeconomic data, the Hodrick and Prescott (1997) filter is often used. In recent years, the Baxter and King (1999) filter and the Christiano and Fitzgerald (2003) filter are also often used. Our method, however, is an alternative to these filters, and it is “structural” estimation of time-varying natural rates<sup>16</sup>. From the mid-1980s to the mid-2000s,  $Y_t^s$  is varying between about 2.0% and 4.0%. It suggests

<sup>13</sup>We remove data from 1980:Q1 to 1980:Q4 to avoid the influences of the second oil shock. The details of the data are described in appendix A.

<sup>14</sup>We use Dynare, developed by Juillard (1996). For the Dynare MCMC estimation, all Japanese data from 1998:Q1 to 1998:Q4 are detrended by the Hodrick-Prescott filter. The prior distributions of the parameters for Dynare are determined following Sugo and Ueda (2008).

<sup>15</sup>We remove the results from 1981:Q1 to 1984:Q4 to avoid the influences of poor prior distributions.

<sup>16</sup>Laubach and Williams (2003) and Trehan and Wu (2007) estimate time-varying equilibrium real interest rate using a simple, backward-looking model of the U.S. economy.

Table 2: Time-varying Parameter Search Space (TVPs' time indices are abbreviated)

Parameter	Search Space
$Y^s$	(0.0, 5)
$\phi_\pi$	(1.0, 3)
$\phi_Y$	(0, 2)
$h, \xi_p, \xi_z, \rho_G, \rho_i$	(0,1)
$\sigma_L, \nu$	positive
$\sigma_C, \sigma_\pi, \sigma_I, \sigma_Q, \sigma_i, \sigma_G, \sigma_Z,$	positive
$\sigma_Y^v, \sigma_C^v, \sigma_\pi^v, \sigma_I^v, \sigma_W^v, \sigma_L^v, \sigma_i^v$	positive
other parameters	no limitation

that the average of the growth of natural output is 3.02 in the long run. From the mid-1980s to the mid-2000s,  $\pi_t^s$  is varying between about 1.5% and 3.5%. The average of  $\pi_t^s$  is about 2.46 from 1985 to 2007. It points that the inflation target rate of the Fed is about 2.5% in the past 20 years. The  $r_t^s$  is an estimate of equilibrium real interest rate. This result shows that an equilibrium real interest rate is negative from the early 2000s to the mid-2000s because the Fed cuts short term nominal interest rates to flight against deflation in the early 2000s.

Figure 2 shows the Investment-Specific Growth (ISG). It indicates that ISG of the US economy slowed down in the late 1980s, and recover from the early 1990s to the early 2000s. The behavior of ISG coincides the booming economy in the 1990s, and is probably related with investment-specific technology changes (IST), proposed by Greenwood et al. (1997). While IST has a microeconomic foundation, our ISG is practically introduced to make the estimation of  $Y^s$  stable because the growth rate of investment is different from the growth rate of natural output. Although our result suggests that ISG has an important roll in the long-run growth, the cause of it is still an open question.

Figure 3 shows the estimates of the endogenous variables. The expectational output gap,  $E_t \hat{Y}_t$ , and the output gap,  $\hat{Y}_t$ , have fundamentally similar shapes. They indicate that the serious recessions happen in 1991. From 1992 to the late 1990s, the output gap is keeping positive. This result shows the booming economy in the Clinton era. From the early 1990s to the late 1990s, the inflation expectation,  $E_t \hat{\pi}_t$ , and the inflation rate,  $\hat{\pi}_t$ , are negative because of disinflation of the US economy. The interest rate,  $\hat{i}_t$ , shows the deviation from the equilibrium real interest rate, and it presents the fact that the Fed cuts short term nominal interest rates in recessions and hikes the rates in the economic booms. The symbol,  $\hat{Z}_t$ , shows the negative technology shocks that happened in the early 1990s, and it correspond to the recessions in 1991.

Fig. 4 shows the estimates of time-varying parameters. These estimates indicate that some "structural" parameters are time-varying. The results indicate that habit persistence,  $h$ , the Calvo parameter,  $\xi_p$ , and the coefficient of AR(1) technology process,  $\xi_z$ , are relatively stable. The parameter,  $\sigma_C$ , is about 1 in most periods, and it indicates that the utility function of consumption is nearly equal to the log-type utility function. The inverse of the elasticity of labor,  $\sigma_L$  is varying from about 2.0 to about 3.5. In table 3, The estimates of  $\sigma_L$ , which are inferred by Levin et al. (2005) (LOWW), Del Negro et al. (2007) (DSSW), Smets and Wouters (2007) (SW), and Hirose and Naganuma (2007) (HN) using the US data, are shown, and they have relatively large standard deviations. These results of previous studies are caused by the fluctuation of time-varying  $\sigma_L$ .

Figure 5 shows the estimates of time-varying parameters of TR. The inertia term,  $\rho_i$ , is from 0.55 to 0.65

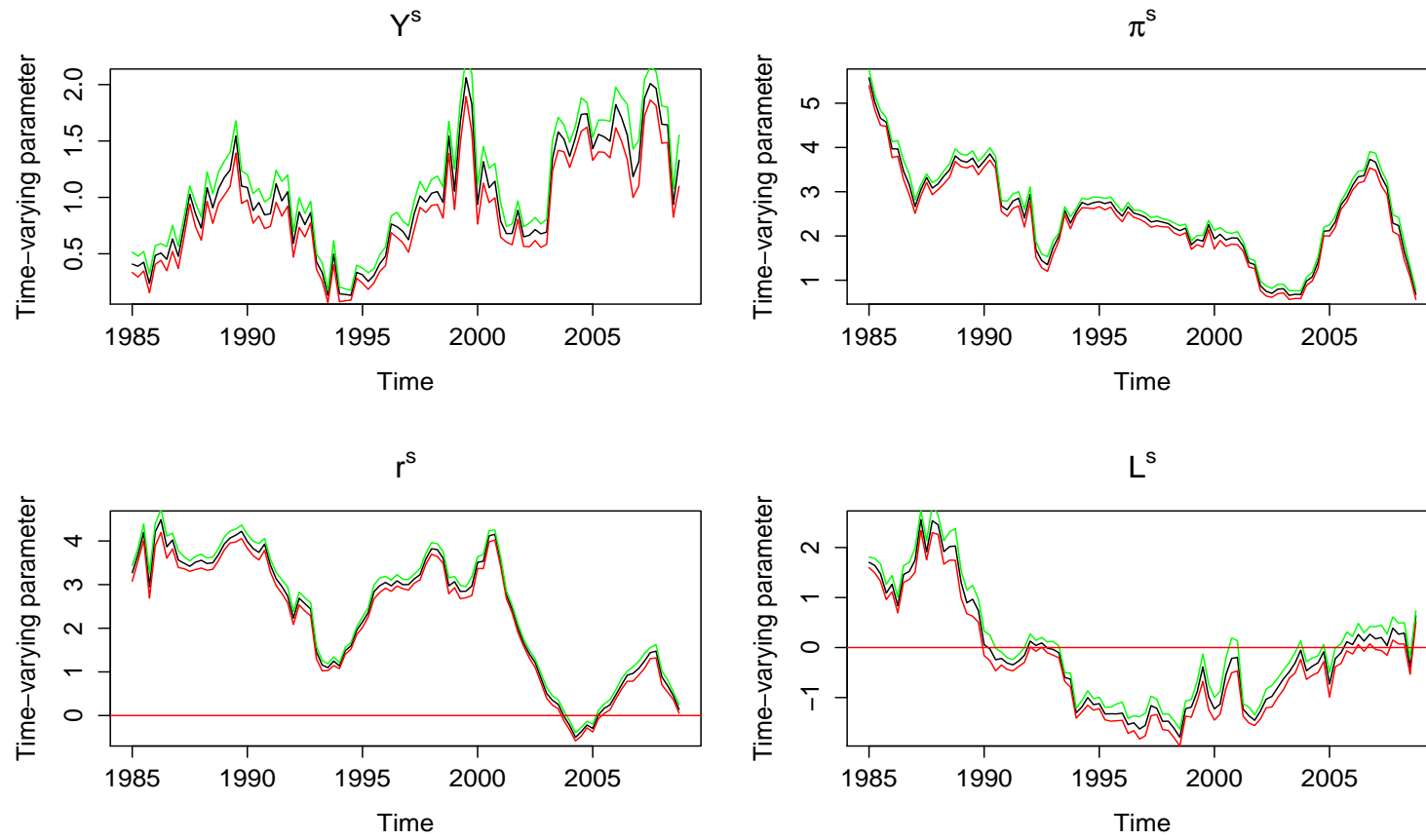


Figure 1: Time-varying natural rates and targets

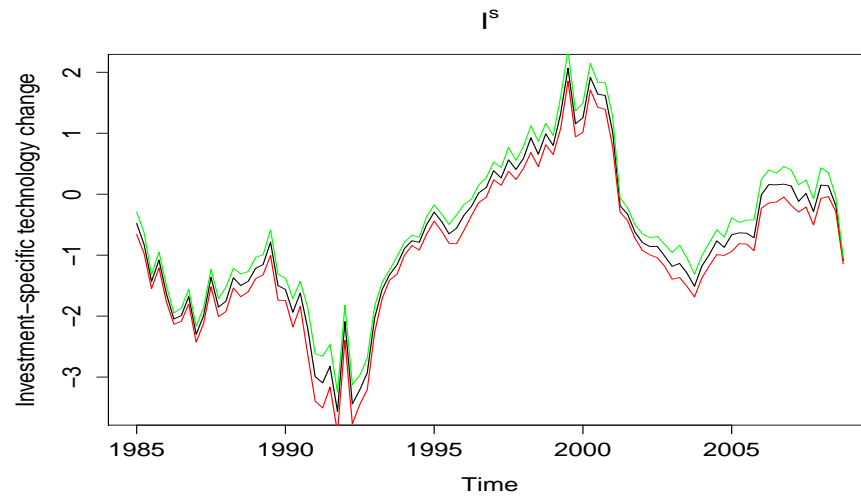


Figure 2: Investment-specific Technology Change

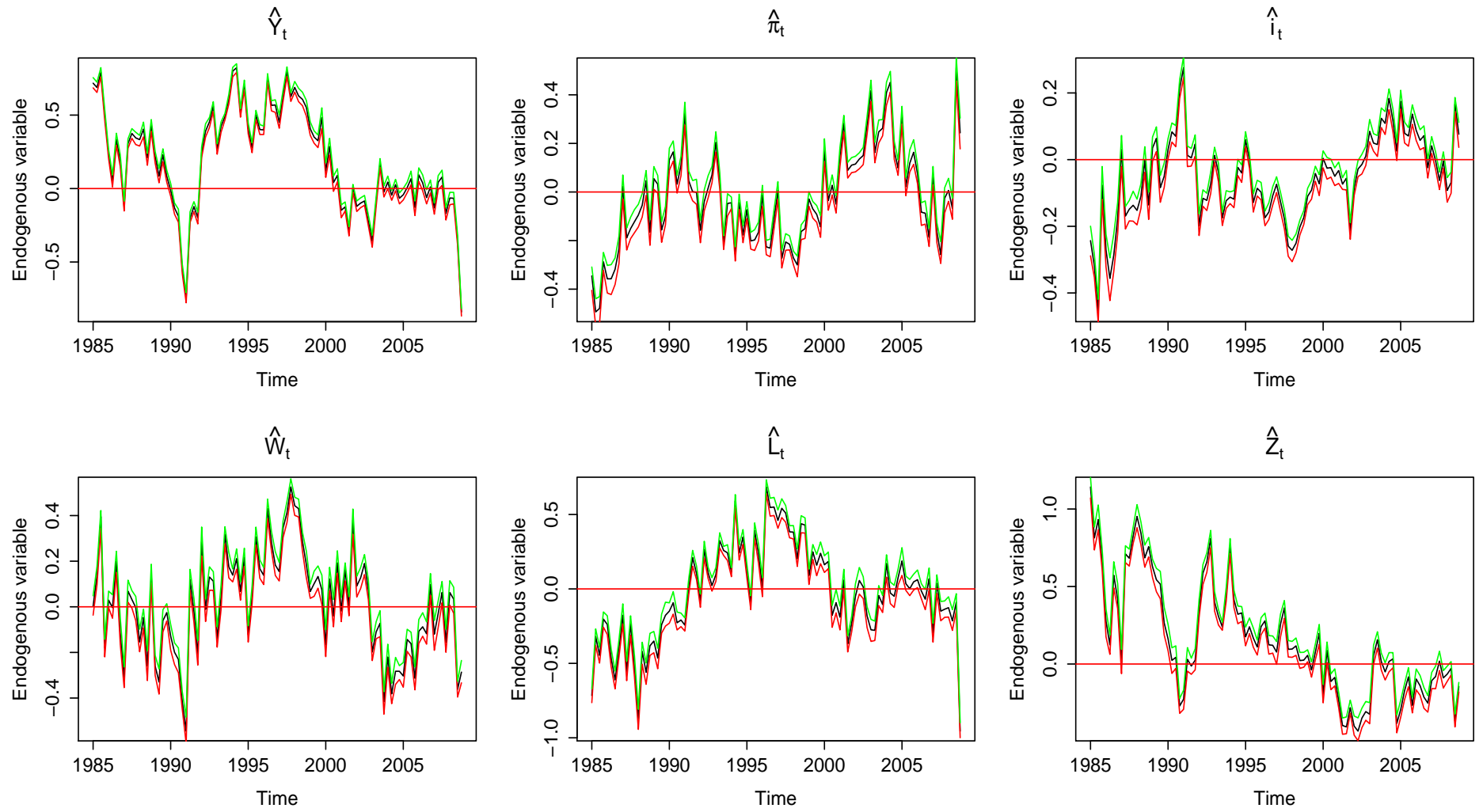


Figure 3: Endogenous variables

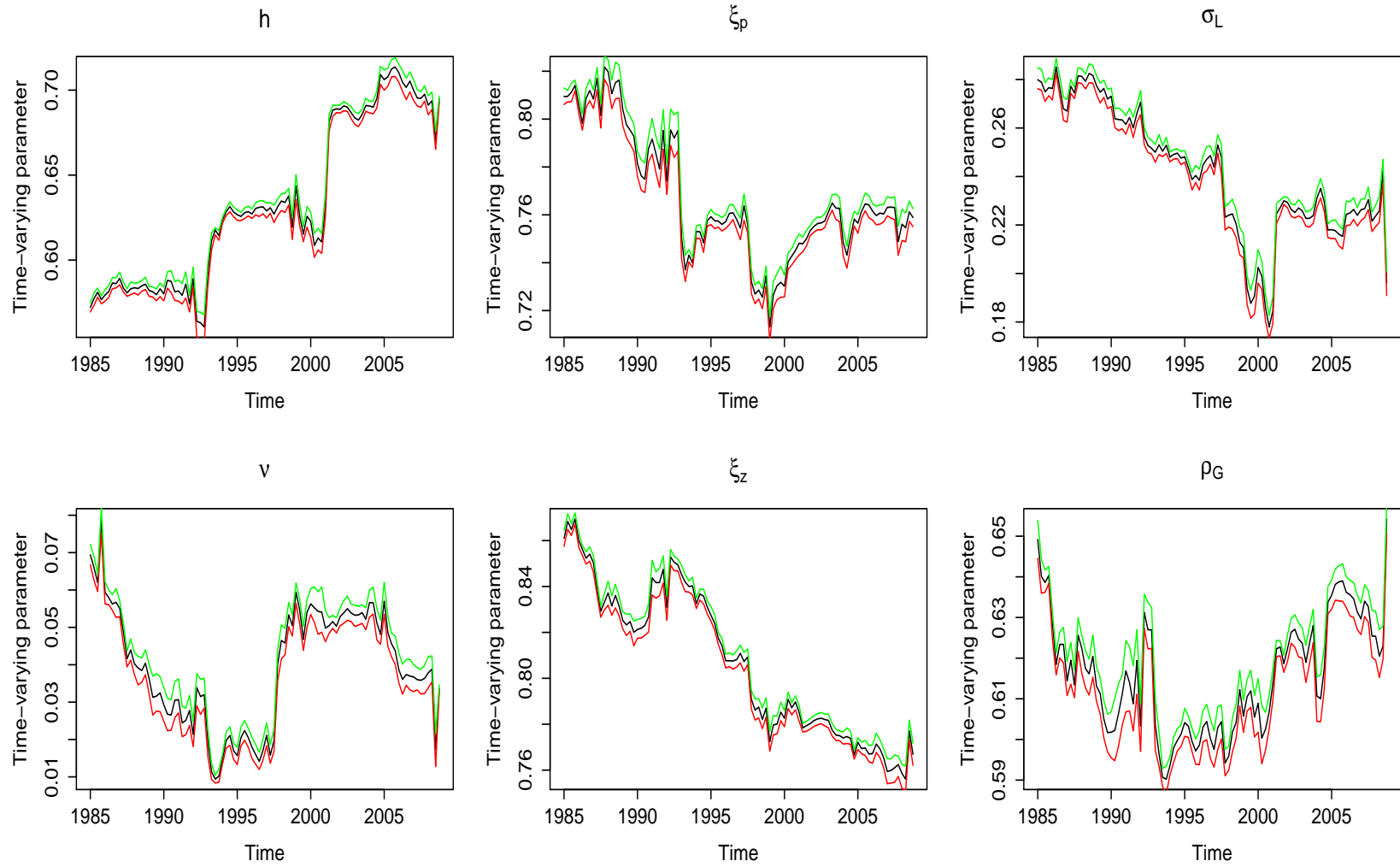


Figure 4: Time-varying parameters

Table 3: Estimates of previous studies

Mean (90% distribution intervals)	LOWW	DSSW	SW	HN
$\sigma_C$	2.19 (1.68-2.74)	-	1.38 (1.16-1.59)	1.75 (1.51-1.97)
$h$	0.29 (0.20-0.38)	0.81 (0.77-0.85)	0.71 (0.64-0.78)	0.59 (0.39-0.79)
$\sigma_L$	1.49 (0.95-2.12)	2.09 (0.95-3.19)	1.83 (0.91-2.78)	0.79 (0.46-1.11)
$\xi_p$	0.83 (0.81-0.86)	0.83 (0.79-0.87)	0.66 (0.56-0.74)	0.82 (0.78-0.86)

in most periods. It indicates that the Fed makes the change of nominal short-term interest rates smooth. The coefficient of the output gap,  $\phi_Y$ , is from 1.8 to 2.1 in most periods. In the 1990s,  $\phi_Y$ , is relatively small rather than in the late 1980s, and it show the change of monetary policy of the Fed. From 1985 to 1990, the coefficient of the inflation rate,  $\phi_\pi$ , increases from 2.5 to 4.0, and this result shows that the the Fed is increasing the focus on the inflation rate from 1985 to the early 1990. From the mid-1990 to present,  $\phi_\pi$  is around 4.0 and very stable. It also indicates that the behaviour of the Fed had changed to realize mild inflation and made it stable from the mid-Greenspan era to the early Bernanke era.

The stochastic volatilities,  $\sigma_{Y,t}$ ,  $\sigma_{\pi,t}$ ,  $\sigma_{i,t}$ ,  $\sigma_{Z,t}$ , are shown in Figure 6. The stochastic volatilities,  $\sigma_{Y,t}^v$ ,  $\sigma_{\pi,t}^v$ ,  $\sigma_{i,t}^v$ , are shown in Figure 7.

In practice, the Hodrick and Prescott (1997) filter and the Baxter and King (1999) filter are often used to estimate the natural output of the US economy. Our method is an alternative to these filters, and it is “structural” estimation of time-varying economic trends. In Figure 8, we compare our annualized estimates of output gap with estimates of the HP filter and the CF filter. In the upper panel of Figure 8, we show our estimate (the black line) and the estimate of the HP filter (the blue line). In the lower panel of Figure 8, we show our estimate (the black line) and the estimate of the CF filter (the green line). The result of the HP filter is not consistent with our result. However, the one of the CF filter coincides with our one.

## 4 Conclusion and Discussion

This paper proposes a new method to estimate parameters, natural rates, and business cycles of dynamic stochastic general equilibrium models simultaneously and consistently. The method is based on the Monte Carlo particle filter, proposed by Kitagawa (1996) and Gordon et al. (1993), and a self-organizing state space model, proposed by Kitagawa (1998). In our method, we estimate the parameters and the natural rates using the time-varying-parameter approach, which is often used to infer invariant parameters practically. In most previous papers on DSGE models, structural parameters of them are assumed to be “deep (invariant).” However, our method analyzes how stable structural parameters are. Our method is based on Bayesian statistics and nonlinear, non-Gaussian, and non-stationary state space modeling to estimate unknown parameters and states. Adopting the TVP approach



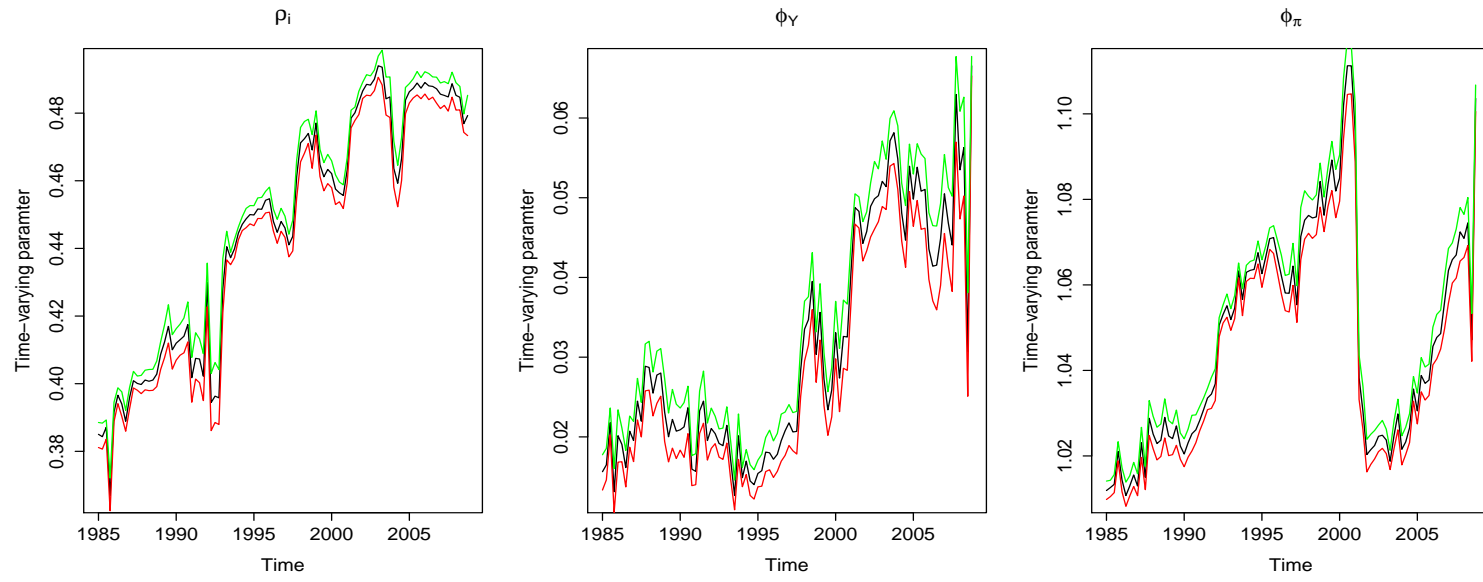


Figure 5: Time-varying parameters of the Taylor rule

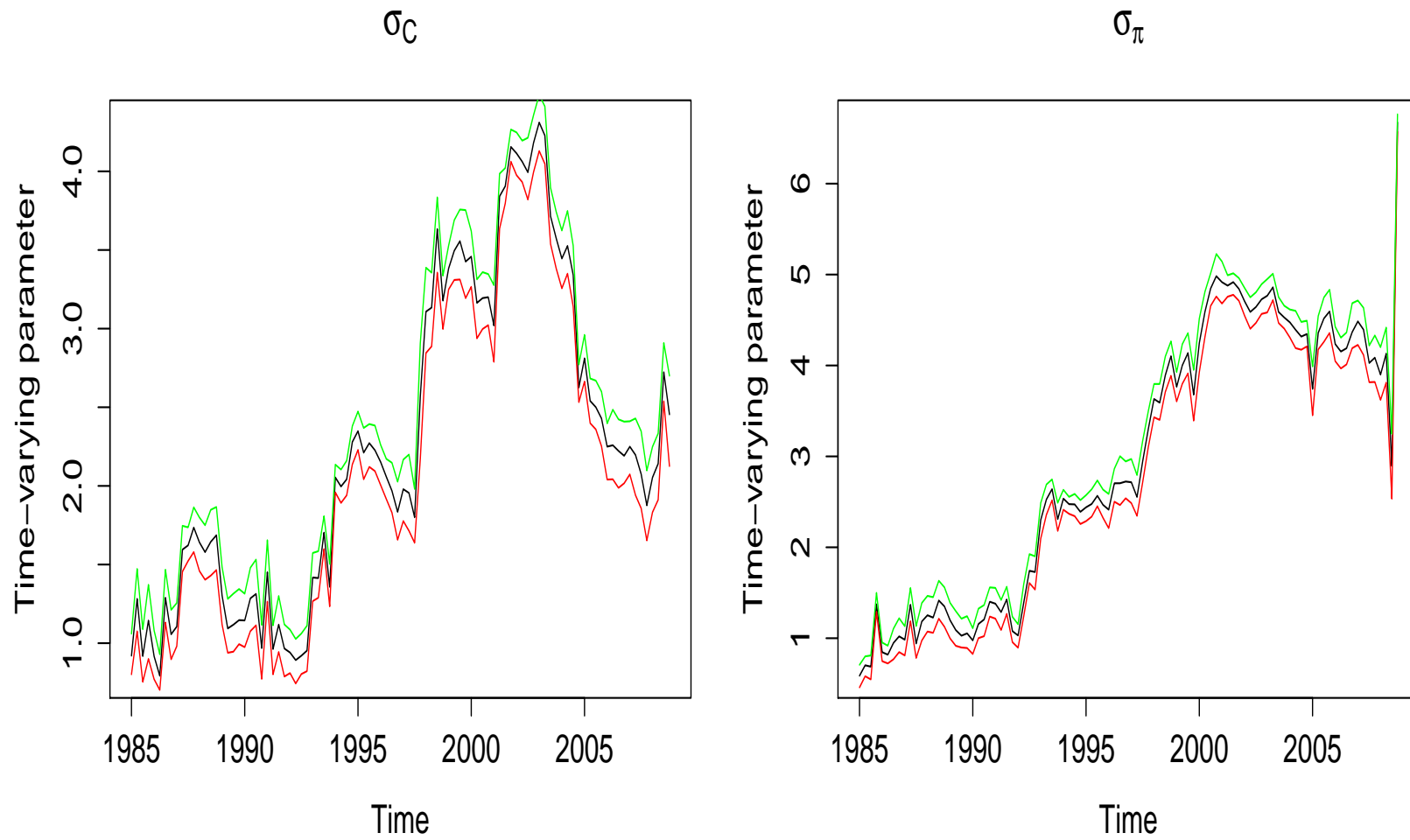


Figure 6: Stochastic Volatilities in the System Equation

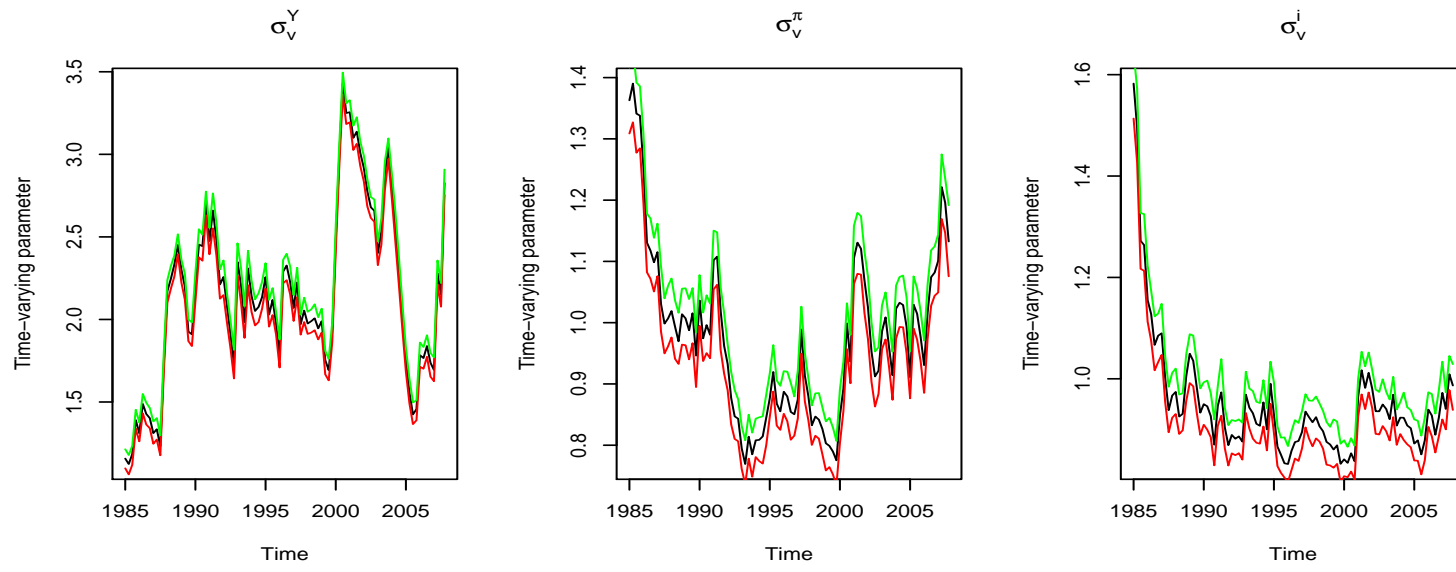
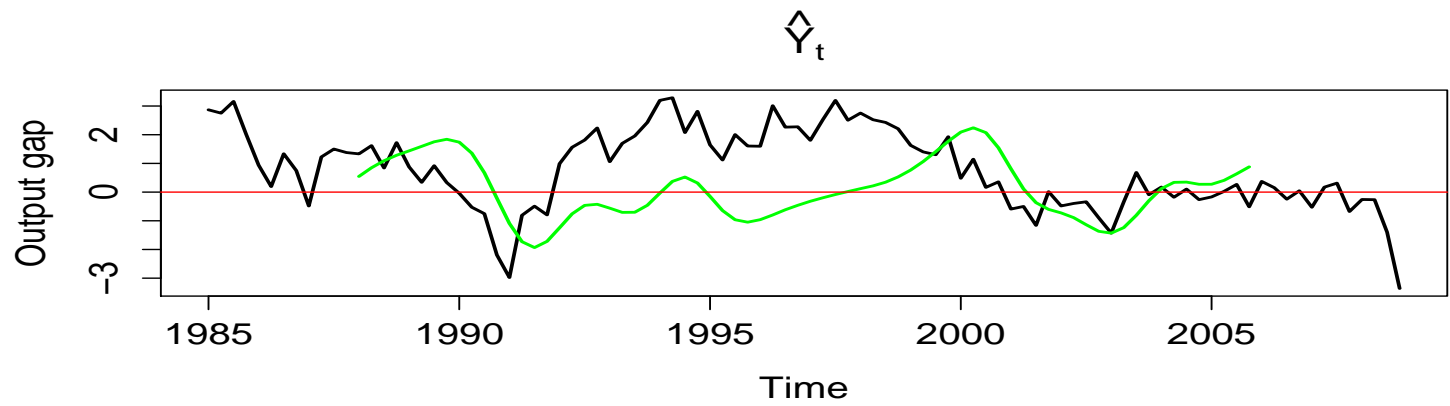
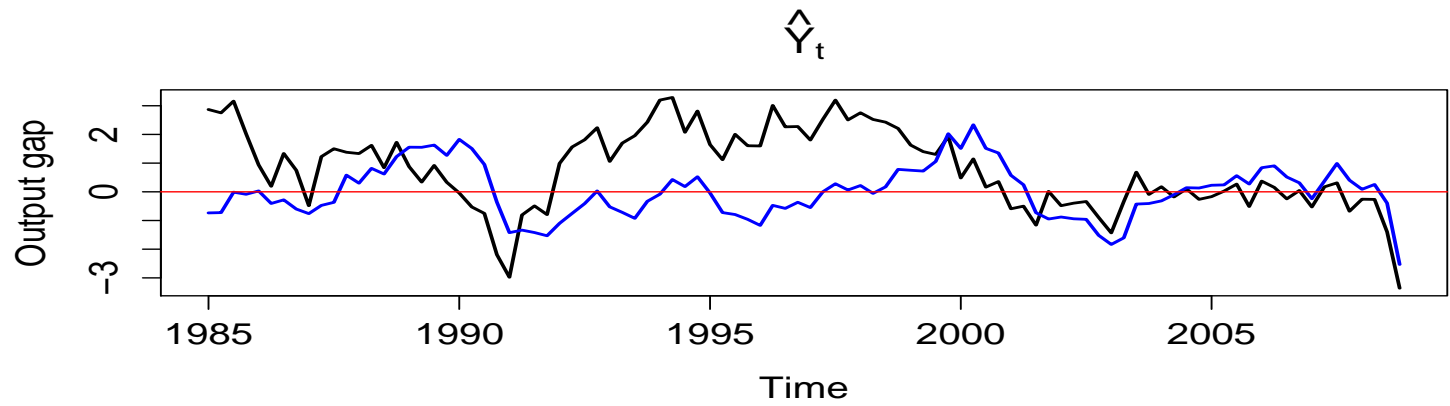


Figure 7: Stochastic Volatilities in the Measurement Equation



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Figure 8: Output gap: Comparing filtering methods

creates the great advantage that the structural changes of parameters are detected naturally. Moreover, we estimate time-varying natural rates of macroeconomic data: real output, inflation rate, and real interest rate. Our method is an alternative to detrending methods based on the Hodrick-Prescott filter, the Baxter-King filter, the Christiano-Fitzgerald filter, and other filtering algorithms.

In empirical analysis, we estimate a new Keynesian DSGE model using the US data. The analysis shows that the average of the growth of natural output is 3.02 and the average of inflation target is 2.46 from 1985 to 2007. It also indicates that the coefficients of the Taylor rule are stable and the Fed focus on the stability of inflation in the Greenspan era. An equilibrium real interest rate is negative from the early 2000s to the mid-2000s because the Fed cut rates to prevent deflation. Our estimate of output gap relatively coincides with the estimates, which are calculated by the HP/CF filters, although our method is totally different from the filters.

In a new study, we are estimating new Keynesian, small open economy DSGE models, new Keynesian DSGE models with liquidity-constraint households, Christiano et al. (2005), and second-order approximation of DSGE models. Furthermore, our method can be easily extended to estimate state-dependent-pricing models with random menu costs, proposed by Dotsey et al. (1999)<sup>17</sup>. In policy analysis of DSGE models, impulse response functions are often used. In our framework, the effectiveness of the traditional way is ambiguous because parameters in DSGE models are time-varying. If we calculate impulse response function at time  $t$ , the results of them may be meaningless because the parameters may have changed at time  $t + 1$ . Canova and Gambetti (2006) proposes the use of generalized impulse response functions in time-varying structural vector autoregressions. However, in time-varying analysis of DSGE models, it is an open question. Additionally, the cause of investment-specific growth is also an open question.

## Appendix A Data Source

We use quarterly macroeconomic data on the US economy from 1981:Q1 to 2007:Q4.

- 3-Month Treasury Bill (Secondary Market Rate, Averages of Business Days), averaged over three months: St. Louis Fed, Economic Data - FRED,
- Seasonally-adjusted Real Gross Domestic Product: St. Louis Fed, Economic Data - FRED
- Seasonally-adjusted GDP deflator: St. Louis Fed, Economic Data - FRED
- Civilian Noninstitutional Population, averaged over three months: St. Louis Fed, Economic Data - FRED
- Seasonally-adjusted Real Private Nonresidential Fixed Investment: St. Louis Fed, Economic Data - FRED
- Seasonally-adjusted Average Weekly Hours: BLS

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<sup>17</sup>Gertler and Leahy (2006) and Bakhshi et al. (2007) derive a Phillips curve equation from a DSGE model with state-dependent pricing.

- Seasonally-adjusted Average Hourly Earnings of Production Workers: BLS

## Appendix B Simulation Setting

The discount factor,  $\beta$ , is calibrated to be 0.99. We use uniform distributions for initial prior distributions of states, time-varying parameters, and parameters:  $uniform(-1, 1)$  for states,  $uniform(0, 1)$  for time-varying parameters, and  $uniform(0, 0.2)$  for parameters. The number of particle is 10,000 at time  $t$ . Thus, we generate 270,000 random variables at time  $t$ . In Eq. (30), we set  $C$  to zero.

## Appendix C Estimation Method

To estimate a state vector  $\mathbf{x}_t$  and a time-varying-parameter vector,  $\tilde{\boldsymbol{\theta}}_t$ , we adopt the Monte Carlo Particle Filter (MCPF), proposed by Kitagawa (1996) and Gordon et al. (1993), and a self-organizing state space model, proposed by Kitagawa (1998).

### Appendix C.1 Nonlinear, Non-Gaussian, and Non-stationary State Space Model

In this subsection, we describe a nonlinear, non-Gaussian, and non-stationary state space model and a self-organizing state space model (MCPF is described in the next subsection).

A nonlinear, non-Gaussian, and non-stationary state space model for the time series  $\mathbf{Y}_t$ ,  $t = \{1, 2, \dots, T\}$  is defined as follows.

$$\begin{aligned}\mathbf{x}_t &= f_t(\mathbf{x}_{t-1}, \boldsymbol{\epsilon}_{1,t}, \boldsymbol{\xi}_s), \\ \mathbf{Y}_t &= h_t(\mathbf{x}_t, \mathbf{v}_t, \boldsymbol{\xi}_o),\end{aligned}\tag{C1}$$

where  $\mathbf{x}_t$  is an unknown  $n_x \times 1$  state vector,  $\boldsymbol{\epsilon}_{1,t}$  is  $n_\epsilon \times 1$  system noise vector with a density function  $q(\boldsymbol{\epsilon}_1|\cdot)$ <sup>18</sup>,  $\mathbf{v}_t$  is  $n_v \times 1$  observation noise vector with a density function  $r(\mathbf{v}|\cdot)$ . The function  $f_t : \mathbf{R}^{n_x} \times \mathbf{R}^{n_\epsilon} \rightarrow \mathbf{R}^{n_x}$  is a possibly nonlinear time-varying function and the function  $h_t : \mathbf{R}^{n_x} \times \mathbf{R}^{n_v} \rightarrow \mathbf{R}^{n_y}$  is a possibly nonlinear time-varying function. The first equation of (C1) is called a system equation and the second equation of (C1) is called an observation equation. We would like to emphasize the functions,  $f_t$  and  $h_t$ , are possibly time dependent. A system equation depends on a possibly unknown  $n_s \times 1$  parameter vector,  $\boldsymbol{\xi}_s$ , and an observation equation depends on a possibly unknown  $n_o \times 1$  parameter vector,  $\boldsymbol{\xi}_o$ . This NNNSS specifies the two following conditional density functions.

$$\begin{aligned}p(\mathbf{x}_t|\mathbf{x}_{t-1}, \boldsymbol{\xi}_s), \\ p(\mathbf{Y}_t|\mathbf{x}_t, \boldsymbol{\xi}_o).\end{aligned}\tag{C2}$$

We define a parameter vector  $\boldsymbol{\theta}$  as follows.

$$\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\xi}_s \\ \boldsymbol{\xi}_o \end{bmatrix}.\tag{C3}$$

We denote that  $\theta_j$ , ( $1 \leq j \leq J$ ) is the  $j$ th element of  $\boldsymbol{\theta}$  and  $J(= n_s + n_o)$  is the number of elements of  $\boldsymbol{\theta}$ . This type of state space model (C1) contains a broad class of linear, nonlinear, Gaussian, or non-Gaussian time series

<sup>18</sup>The system noise vector is independent of past states and current states.

models. In state space modeling, estimating the state space vector  $\mathbf{x}_t$  is the most important problem. For the linear Gaussian state space model, the Kalman filter, which is proposed by Kalman (1960), is the most popular algorithm to estimate the state vector  $\mathbf{x}_t$ . For nonlinear or non-Gaussian state space models, there are many algorithms. For example, the extended Kalman filter (Jazwinski (1970)) is the most popular algorithm; other examples are the Gaussian-sum filter (Alspach and Sorenson (1972)), the dynamic generalized model (West et al. (1985)), and the non-Gaussian filter and smoother (Kitagawa (1987)). In recent years, MCPF for NNNSS has been a popular algorithm because it is easily applicable to various time series models <sup>19</sup>.

In econometric analysis, generally, we don't know the parameter vector  $\boldsymbol{\theta}$ . In our framework, the unknown parameter vectors are  $\boldsymbol{\xi}_o$  and  $\boldsymbol{\xi}_s$  <sup>20</sup>. In traditional parameter estimation, maximizing the log-likelihood function of  $\boldsymbol{\theta}$  is often used. The log-likelihood of  $\boldsymbol{\theta}$  in MCPF is proposed by Kitagawa (1996). However, MCPF is problematic to estimate the parameter vector  $\boldsymbol{\theta}$  because the likelihood of the filter contains errors from the Monte Carlo method. Thus, you cannot use nonlinear optimizing algorithm like Newton's method <sup>21</sup>. To solve the problem, Kitagawa (1998) proposes a self-organizing state space model. In Kitagawa (1998), an augmented state vector is defined as follows.

$$\mathbf{z}_t = \begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{\Theta}_t \end{bmatrix}, \quad (\text{C4})$$

where  $\boldsymbol{\Theta}_t = (\tilde{\boldsymbol{\theta}}_t, \boldsymbol{\theta})^t$ ,  $\tilde{\boldsymbol{\theta}}_t$  is a vector of time-varying parameters, and  $\boldsymbol{\theta}$  is a vector of invariant parameters. Note that  $\tilde{\boldsymbol{\theta}}_t = \tilde{\boldsymbol{\theta}}_{t-1} + \boldsymbol{\epsilon}_{2,t}$ , with  $\boldsymbol{\epsilon}_{2,t}$  a white noise sequence distributed with a density function  $p_2(\boldsymbol{\epsilon}_{2,t} | \boldsymbol{\Sigma}_{\boldsymbol{\xi}_s})$ . An augmented system equation and an augmented measurement equation are defined as

$$\begin{aligned} \mathbf{z}_t &= F_t(\mathbf{z}_{t-1}, \boldsymbol{\epsilon}_t, \boldsymbol{\xi}_s), \\ \mathbf{Y}_t &= H_t(\mathbf{z}_t, \mathbf{v}_t, \boldsymbol{\xi}_o), \end{aligned} \quad (\text{C5})$$

where

$$F_t(\mathbf{z}_{t-1}, \boldsymbol{\epsilon}_t, \boldsymbol{\xi}_s) = \begin{bmatrix} f_t(\mathbf{x}_{t-1}, \boldsymbol{\epsilon}_{1,t}, \boldsymbol{\xi}_s) \\ \tilde{\boldsymbol{\theta}}_{t-1} + \boldsymbol{\epsilon}_{2,t} \\ \boldsymbol{\theta} \end{bmatrix}$$

and

$$H_t(\mathbf{z}_t, \mathbf{v}_t, \boldsymbol{\xi}_o) = h_t(\mathbf{x}_t, \mathbf{v}_t, \boldsymbol{\xi}_o)$$

where  $\boldsymbol{\epsilon}_t = (\boldsymbol{\epsilon}_{1,t}, \boldsymbol{\epsilon}_{2,t})^t$ . This NNNSS is called a self-organizing state space (SOSS) model. In our method, we stress that states, time-varying parameters, and invariant parameters are estimated simultaneously. Therefore, our problem is how to estimate  $\mathbf{z}_t$ .

<sup>19</sup>Many applications are shown in Doucet et al., eds (2001).

<sup>20</sup>Details of  $\boldsymbol{\xi}_o$  and  $\boldsymbol{\xi}_s$  are discussed in the next subsection.

<sup>21</sup>See Yano (2008a).

## Appendix C.2 Monte Carlo Particle Filter

The Monte Carlo particle filter is a variant of sequential Monte Carlo algorithms. In MCPF, the expectation of a posterior distribution are approximated using “particles” that have weights.

$$E[p(\mathbf{z}_t|\mathbf{Y}_{1:t})] \simeq \frac{1}{\sum_{m=1}^M w_t^m} \sum_{m=1}^M w_t^m \delta(\mathbf{z}_t - \mathbf{z}_t^m), \quad (\text{C6})$$

where  $w_t^m$  is the weight of a particle  $\mathbf{z}_t^m$ ,  $M$  is the number of particles, and  $\delta$  is the Dirac’s delta function <sup>22</sup>. Weights  $w_t^m$   $m = \{1, 2, \dots, M\}$  are defined as follows.

$$w_t^m = r(\psi(\mathbf{y}_t, \mathbf{z}_t^m)) \left| \frac{\partial \psi}{\partial \mathbf{y}_t} \right|, \quad (\text{C7})$$

where  $\psi$  is the inverse function of the function  $h$  <sup>23</sup>. The right hand side of Eq. (C7) is the likelihood function of an NNNSS model. In the standard algorithm of MCPF, the particles  $\mathbf{x}_t^m$  are resampled with sampling probabilities proportional to  $w_t^1, \dots, w_t^M$ . Resampling algorithms are discussed in Kitagawa (1996). After resampling, we have  $w_t^m = 1/M$ . Therefore, Eq. (C6) is rewritten as

$$E[p(\mathbf{z}_t|\mathbf{Y}_{1:t})] \simeq \frac{1}{M} \sum_{m=1}^M \delta(\mathbf{z}_t - \hat{\mathbf{z}}_t^m), \quad (\text{C8})$$

where  $\hat{\mathbf{z}}_t^m$  are particles after resampling. Particles  $\mathbf{x}_t^m$   $m = \{1, 2, \dots, M\}$  are sampled from a system equation:

$$\mathbf{z}_t^m \sim p(\mathbf{z}_t | \mathbf{z}_{t-1}^m, \boldsymbol{\xi}_s). \quad (\text{C9})$$

Kitagawa (1996) shows that the log-likelihood of  $\boldsymbol{\theta}$  is approximated by

$$l(\boldsymbol{\theta}) \simeq \sum_{t=1}^T \log\left(\sum_{m=1}^M w_t^m\right) - T \log M, \quad (\text{C10})$$

where  $T$  is the number of observations. Using Eq. (C10), we can compare the fits of DSGE models. In self-organizing state space modeling, the augmented state vector is estimated using MCPF. Thus, states and parameters are estimated simultaneously without maximizing the log-likelihood of Eq. (C5) because the parameter vector  $\boldsymbol{\theta}$  in Eq. (C5) is approximated by particles and it is estimated as the state vector in Eq. (C4) <sup>24</sup>.

On a self-organizing state space model, however, Hürseler and Künsch (2001) points out a problem: determination of initial distributions of parameters for a self-organizing state space model. The estimated parameters of a self-organizing state space model comprise a subset of the initial distributions of parameters. We must know the posterior distributions of parameters to estimate parameters adequately. However, the posterior distributions of the

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<sup>22</sup>The Dirac delta function is defined as

$$\begin{aligned} \delta(x) &= 0, \text{ if } x \neq 0, \\ \int_{-\infty}^{\infty} \delta(x) dx &= 1. \end{aligned}$$

<sup>23</sup>See Kitagawa (1996).

<sup>24</sup>The justification of an SOSS model is described in Kitagawa (1998).



parameters are generally unknown. Parameter estimation fails if we do not know their appropriate initial distributions. Yano (2008a) proposes a method to seek initial distributions of parameters for a self-organizing state space model using the simplex Nelder-Mead algorithm to solve the problem. In this paper, we use uniform distributions for initial distributions of time-varying parameters because most time-varying parameters are restricted to be more than zero and less than unity.

### Appendix C.3 Time-varying Parameters

In this paper, we assume the “symmetric” random walk prior (the Litterman prior) to estimate time-varying parameters (see Doan et al. (1984))<sup>25</sup>. The random walk prior is given by

$$\tilde{\theta}_t = \tilde{\theta}_{t-1} + \epsilon_{2,t}, \quad (\text{C11})$$

where  $\epsilon_{2,t} \sim q(\epsilon_{2,t}|\Sigma_{\xi_s})$ ,  $q(\epsilon_{2,t}|\Sigma_{\xi_s})$  is a Gaussian distribution, and  $\Sigma_{\xi_s}$  is a diagonal matrix. In general, the diagonal components,  $\{\xi_{1,s}, \xi_{2,s}, \dots, \xi_{L,s}\}$ , of  $\Sigma_{\xi_s}$  are different. In this paper, however, to reduce computational complexity, we define time evolution of a coefficient as follows:

$$\tilde{\theta}_{i,t} = \tilde{\theta}_{i,t-1} + |\xi_{s,\cdot}| \epsilon_{2,i,t}, \quad (\text{C12})$$

where  $\epsilon_{2,i,t} \sim N(0, |\xi_{s,2}|)$  if  $h, \xi_p, \sigma_L, \nu, \xi_Z, \Psi_C, \Psi_I, \Psi_G, \phi_Y, \phi_\pi$ , and  $\rho_G$  and  $\epsilon_{2,i,t} \sim |\xi_{s,1}| \times t(df = 25)$  if otherwise. Note that  $\sigma_{L,t}, \xi_{Z,t}, \phi_{Y,t}, \phi_{\pi,t}, \sigma_{Y,t}, \sigma_{\pi,t}, \sigma_{i,t}, \sigma_{Z,t}, \sigma_{Y,t}^v, \sigma_{\pi,t}^v, \sigma_{i,t}^v$  are restricted to be positive and  $h_t, \sigma_{C,t}, \xi_{p,t}$ , and  $\rho_{i,t}$  are restricted to be more than zero and less than unity. The particles that violate these restrictions are numerically discarded before resampling.

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<sup>25</sup>See also a traditional approach, proposed by Cooley and Prescott (1976). The smoothness priors proposed by Kitagawa (1983) is a generalization of the random walk priors.

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