On Local Polynomial Estimation of State-Price Densities: An Application to Japanese Option Pricing

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Abstract. Following the method proposed by Aït-Sahalia and Lo [2], this paper conducts a nonparametric estimation of state-price densities implicit in the Osaka stock index option prices. This paper offers two implications. First, the local polynomial estimator works better for data sets such as ours, whose observations are sparse near the boundaries and unevenly distributed in the interior, than the Nadaraya-Watson estimator, most frequently used in existing literature. Second, a structural break is reasonably identified for state price densities at the mid-May/last-May or the last-May/first-June of year 2003 according to the local polynomial estimator.
1. Introduction  In modern financial economics, state-price density plays an essential role in pricing financial assets. Asset prices are equal to the expected future payoff, which is calculated according to the state-price density. This important building block for asset pricing is the product of the physical density that assigns a probability measure to each state, and the price that risk-averse investors are willing to pay in order to obtain one unit of goods in the corresponding state.

Following the method proposed by Aït-Sahalia and Lo [2], this paper conducts a non-parametric estimation of the state-price density implicit in the Osaka stock index option prices. Ross [19], Banz and Miller [3], Breeden and Litzenberger [5], and others find that the state-price density is closely related to the option price; that is to say, the second derivative of the call-pricing function with respect to the exercise price must equal the state-price density given a state of stock prices. Exploiting this elegant finding, Aït-Sahalia and Lo successfully calculate the state-price density from nonparametrically estimated option pricing functions, using data obtained from the S&P 500 Index Options traded at the Chicago Board Options Exchange.

Our study is motivated by the empirical environment where the pricing data available from the Osaka stock index options are rather sparse near boundaries and highly clustered in the interior of exercise prices. The price data are also unevenly distributed in terms of (inverse) moneyness. Such a feature of our data set reflects both fewer points of exercise prices and less terms of maturity for the option contracts listed at the Osaka Securities Exchange. Given a length of a sample period, the data points provided by the Osaka
stock index options are approximately one third as many as those from the S&P 500 Index Options. In addition, it is not be possible to pool observations for a longer sample period, for example one year as in Aït-Sahalia and Lo [2], if the state-price density is not so stable over time in the case of the Osaka Securities Exchange.

In the above sense, the characteristics of our data set contrast sharply with those of Aït-Sahalia and Lo [2]. If rich data points are available as in Aït-Sahalia and Lo [2], then the Nadaraya-Watson kernel estimator, one of the most popular methods and that adopted by Aït-Sahalia and Lo, has desirable features in terms of estimation. The Nadaraya-Watson estimator, however, does not work well for non-uniformly distributed data set such as ours. As discussed in Ruppert and Wand [20] and others, the Nadaraya-Watson estimator tends to yield serious biases in terms of behavior near boundaries, while as Fan [8] points out, it does not perform well for data sets whose observations are highly clustered in the interior. As those authors propose, the local linear kernel-weighted least squares regression estimator (hereafter, the local polynomial estimator) is expected to perform better in the case of non-uniformly distributed data sets. In particular, it is possible to estimate precisely derivatives of approximated functions. Given our non-uniformly distributed data and interest in the second derivative of option pricing functions, the local polynomial estimator is fairly desirable for our purpose of estimation.

The main difference between the Nadaraya-Watson estimator and the local polynomial estimator is that each point of a certain function is locally approximated by only a constant term in the former, while it is approximated by not only a constant term, but also a
polynomial in the latter. The former type of approximation requires enough observations on both sides of each point of explanatory variables; consequently, the Nadaraya-Watson estimator tends to yield biases for either near-boundaries or unevenly-distributed data sets. Thanks to the latter type of approximation, on the other hand, the local polynomial estimator is design-adaptive in that the order of the asymptotic bias does not depend heavily on how unevenly data points are distributed (see Fan [8]). In addition, this estimator allows us to obtain precise estimates of derivatives of an approximated function from the estimated coefficients of polynomials.

The application of the local polynomial estimator to economic data has been growing. Some recent examples include Aït-Sahalia and Duarte [1], Härdle and Tsybakov [11], Heckman, Ichimura, and Todd [12], and Wilson and Carey [23]. In particular, Aït-Sahalia and Duarte [1] apply the local polynomial method to the estimation of state-price densities. Their focus is the development of the nonparametric estimation method that imposes some theoretical restrictions to the option pricing function, and the improvement of the local polynomial method. As discussed later, on the other hand, one of our purposes is to examine whether the theoretical advantage of the local polynomial method, that is the design adaptiveness, is indeed meaningful to actual empirical analyses. We attempt to demonstrate the empirical advantage of the local polynomial method in the context of realistic situations where sparse and inhomogeneous density do matter. In this sense, our research and Aït-Sahalia and Duarte [1] are complementary to each other.

This paper carries out two purposes through the empirical exercise. First, we demon-
strate potential applicability of the local polynomial estimator in estimating state-price density functions from data sets with fewer data points due to either a limited number of listed contracts or a shorter sample period. Second, we apply the local polynomial estimator to estimate state price densities implied by the stock pricing of year 2003 when the Japanese stock market experienced a dramatic changes at the middle of the year. That is, as Figure 1 demonstrates, stock prices measured in terms of the Nikkei 225 Index almost stagnated up to May, however they had moved upward since June. By conducting a stability test, we test whether a structural break point can be identified in state price densities for the corresponding period according to the local polynomial estimator.

The paper is organized as follows. Section 2 briefly describes the derivation of state-price density, while Section 3 reports the estimation methods as well as the test method of stability. Section 4 discusses the estimation results together with stability tests. Section 5 presents a conclusion.

2. Derivation of State-Price Densities In one of the most elegant studies in the literature on asset pricing, Ross [19], Banz and Miller [3], Breeden and Litzenberger [5], and others discover a close relationship between European-type option pricing and state-price density $f_t(S_T)$ as follows:

$$f_t(S_T) = \exp(\tau r_{t,\tau}) \frac{\partial^2 H(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau})}{\partial X^2} \bigg|_{X=S_T}, \quad (1)$$

where $H$ is the date-$t$ pricing function of a call option maturing at date $T$ ($\equiv t + \tau$) with an exercise price $X$ when a date-$t$ stock price is $S_t$. $r_{t,\tau}$ is the risk-free rate between dates $t$
and $T$, while $\delta_{t,\tau}$ is the dividend yield during the same interval. This relation implies that it is possible to derive state-price densities as soon as option pricing functions are available.

Financial economists have been prevented from fully exploiting this theoretical finding by their limited knowledge of parametric forms of option pricing functions. The famous Black-Scholes/Merton option pricing formula (Black and Scholes [4] and Merton [16]), devised in the early 1970s, is still one of only a few parametric examples. To overcome this difficulty, Aït-Sahalia and Lo [2] propose to calculate state-price densities from nonparametrically estimated option pricing functions.

To implement this technique, Aït-Sahalia and Lo make some practical assumptions to reduce the number of regressors in the function $H$ so that typical nonparametric estimation methods may be adapted to the purpose of investigating state-price densities. First, they assume that both the current asset price $S_t$ and the dividend yield $\delta_{t,\tau}$ affect option pricing functions only through the theoretical futures price based on the cost-of-carry model or $F_{t,\tau} = \exp(\tau(r_{t,\tau} - \delta_{t,\tau}))S_t$. Second, the instantaneous risk-free rate is supposed to be constant between dates $t$ and $T$, and $r_{t,\tau}$ is then replaced by $r$. Thanks to these simplifying assumptions, the function $H(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau})$ reduces to $H(F_{t,\tau}, X, \tau)$ given the constant instantaneous risk-free rate $r$.

We further assume that the option pricing function is homogeneous to degree one with respect to inverse moneyness ($M_{t,\tau} \equiv X/F_{t,\tau}$). As a result, there are two regressors, inverse moneyness ($M_{t,\tau}$) and time-to-maturity ($\tau$). To nest the Black-Scholes/Merton option pricing formula within our specification, the option prices are divided by futures prices.
Using the normalized option prices, the Black-Scholes/Merton option pricing formula can be rewritten as a function of $M_{t,\tau}$ and $\tau$.

$$H_t^* \equiv \frac{H_t}{F_{t,\tau}} \equiv e^{-r\tau} \Phi \left( \frac{\ln(F_{t,\tau}/X) + 0.5\sigma^2\tau}{\sqrt{\sigma^2\tau}} \right) - \frac{X}{F_{t,\tau}} e^{-r\tau} \Phi \left( \frac{\ln(F_{t,\tau}/X) - 0.5\sigma^2\tau}{\sqrt{\sigma^2\tau}} \right)$$

$$= e^{-r\tau} \Phi \left( \frac{-\ln(M_{t,\tau}) + 0.5\sigma^2\tau}{\sqrt{\sigma^2\tau}} \right) - M_{t,\tau} e^{-r\tau} \Phi \left( \frac{-\ln(M_{t,\tau}) - 0.5\sigma^2\tau}{\sqrt{\sigma^2\tau}} \right).$$

where $H$ denotes the European-type call option price.

In addition, Aït-Sahalia and Lo pay particular attention to data construction in the following respects. First, the option price may be observed at a different time from either the futures price or the underlying stock index price, since index options, index futures, and individual stocks are usually traded in different exchanges. In the US, for example, the S&P 500 index futures are traded on the Chicago Mercantile Exchange, while the corresponding index options are transacted on the Chicago Board Options Exchange.

Second, option prices are seriously distorted when they are deep in the money (hereafter, ITM) because deep ITM options are likely to be rather illiquid. Consequently, the put-call parity relation does not hold in strict equality. That is, the following equality may not obtain:

$$H + X \exp(-\tau r) = G + F_{t,\tau} \exp(-\tau r),$$

where $H$ ($G$) denotes the European-type call (put) option price.

Aït-Sahalia and Lo propose using the at-the-money put-call parity to avoid these two problems. Given both call and put contracts with exercise prices close to current index
prices (at-the-money contracts), the futures price $F_{t,\tau}$ can be inferred from $\exp(r\tau)(H - G + X \exp(-r\tau))$. Thus, the futures price inferred without any reference to either futures prices or stock index prices is perfectly free from the timing problem. Using this inferred futures price and the out-of-the-money (hereafter, OTM) put option price, the ITM call option price can be inferred from $G + (F_{t,\tau} - X) \exp(-\tau r)$. Thus, the inferred ITM call option price can avoid the above-mentioned price distortion. In doing so, they estimate state-price densities such that the estimation results may not be influenced by market illiquidity. Ait-Sahalia and Lo employ the most popular estimation method, namely, the Nadaraya-Watson kernel estimator.

3. Estimation and Test Methods

3.1. Option transaction at the Osaka Securities Exchange In 1989, the Osaka Securities Exchange (OSE) began to list the option contracts written on the Nikkei stock index, the most popular stock index in Japan\footnote{Although the OSE has become a central market for stock index options in Japan, there have been few academic investigations into the OSE option pricing. One study, Nishina and Nabil [18], examines both the time-series nature of the OSE option pricing and the relevance of the put-call parity condition. Nakamura and Shiratsuka [17] investigate the dynamic change in state-price densities approximated by a simple finite difference method. While the latter research is related to our investigation, it does not employ a rigorous statistical method in estimating state-price densities.}. The OSE initially employed the American-type exercise method. Consequently, as predicted by theory, early exercises were often observed among put option contracts. Unfortunately, a close theoretical relation for the state-price density such as that represented by equation (1) cannot be demonstrated for American-type option contracts.

The OSE, however, switched from the American-type exercise method to the European-
type for contracts maturing later than May 1992\(^2\). This institutional change at the OSE allows us to investigate thoroughly the theoretical implications of equation (1) using Osaka stock index option prices. Our empirical investigation of state-price density thus exploits cross-sectional data obtained from OSE option prices since the second half of 1992.

At the OSE, the exercise prices with an interval of 500 yen were set at seven points around the spot Nikkei index price in 1992, and have been at five or six points since 1993. With regard to the exercise expiration, option contracts are traded in four near-term months. Given this set of listed option contracts, the number of call-put price pairs available for a half-a-year sample period on the closing price basis is around 3500 for the year 1992, while it ranges from 2500 to 5000 for the period after. If the pricing observations based on inactive transactions are excluded, then the sample sizes are even smaller.

As documented for US stock index options, the pricing of ITM contracts traded at the OSE is distorted seriously due to illiquid markets. Because the Osaka stock index options are traded in an auction manner and are not mediated by dealers, bid-ask spreads cannot be used as the measure of market liquidity. Other pieces of evidence, however, suggest that market illiquidity distorts option pricing. As shown by Figures 2-1 and 2-2 based on the 1993 transaction, option trades are concentrated in OTM call and put contracts, that is, for call contracts with moneyness \((S_t/X)\) less than one and put contracts with moneyness larger than one.

Based on the sample period of the second half of 1992, Figure 3 plots the deviation from

\(^2\) The switch from the American-type exercise method to the European mainly reflected a strong request from the OSE members who wanted to spare their customers high administrative costs.
the put-call parity condition, or \( H - G + (X - F_{t,\tau}) \exp(-\tau r) \) against moneyness. As the first panel shows, the deviation from the parity condition is serious for many pairs of call and put contracts; however, as shown in the second panel, \( H - G + (X - F_{t,\tau}) \exp(-\tau r) \) is close to zero for most pairs without any case where the number of a one-day transaction for either put or call is below fifty. In other words, the illiquid trading of ITM contracts is largely responsible for the deviation from the parity condition.

More precisely, the ITM option pricing is distorted downward relative to the OTM option pricing for contracts with maturity longer than one month. That is, the value of \( H - G + (X - F_{t,\tau}) \exp(-\tau r) \) is negative in the case of deep ITM call contracts, while it is positive in the case of deep ITM put contracts. Deep ITM contracts close to the expiration, on the other hand, are quite expensive with large implied volatility, although such contracts are traded infrequently. Figure 4 draws the smoothed shape of the implied volatility against time-to-maturity using the sample from the second half of 1992, and clearly indicates the implied volatility is extremely large as contracts are close to the expiration. Again, deep ITM contracts fairly close to maturity seem to be priced irregularly due to illiquid markets.

The above illiquid features of the ITM option contracts have not changed at all. Given those pieces of evidence, we will not use any pricing information concerning ITM contracts, and will infer ITM option prices from OTM prices based on the put-call parity condition. Like in the case of Aït-Sahalia and Lo [2], such data construction prevents the estimation results from being influenced by the pricing distortion due to illiquid markets.
3.2. Data construction  As discussed in the introduction, this paper uses cross-sectional data obtained from the OSE option prices for the sample period of 2003 when the Japanese stock market experienced a dramatic change at the mid-year. In the data set used for estimation, option prices are recorded for the closing transaction of each option contract.

Given the features of the OSE option pricing discussed in the previous subsection, in particular the distortion of ITM option pricing, this paper follows the data construction method adopted by Aït-Sahalia and Lo [2]. First, futures prices are inferred from the put-call parity condition for contracts closest to at-the-market. Second, given both the inferred futures prices and the observed market prices of OTM put options, ITM call option prices are constructed from the parity condition. For both procedures, we adopt the rate on the certificate of deposit (CD rate) as money market rates. In terms of the former treatment, however, even if the actual futures prices are used together with the observed money market rates, the estimation results do not differ substantially from those with the inferred futures prices. Finally, we exclude not only the contracts that mature in less than five days, but also the pricing data based on less than fifty transactions in order to further control the effect of illiquid markets. Because of the latter type of omission, longer term contracts are often removed. In addition, as an alternative dataset, we pool only observations whose maturity is between one week and four weeks to reduce the dimension of arguments in an option pricing function by removing the remaining periods up to maturity ($\tau$).

The above data construction yields at most about 4000 observations for the first half of 2003. Compared with the case of the Chicago Board Options Exchange (CBOE), analyzed
by Aït-Sahalia and Lo [2], the sample size is rather small in our case. In the CBOE, option contracts are traded in six terms of maturity with more than ten points of exercise prices. Consequently, around 8500 observations are available for a half-a-year sample period. Because of this difference in how option contracts are traded, the observations recorded in our data set are more sparse near the boundaries of exercise prices, and more unevenly distributed in their interior. It is due to this characteristic that our data set invites special consideration for estimation procedures.

3.3. Two estimation methods As mentioned in the Introduction, this study compares two different methods for the nonparametric estimation of state-price densities, the Nadaraya-Watson estimator and the local polynomial estimator. This subsection briefly describes both methods.

As the following argument shows, the major difference between these two estimators is how each point of the pricing function concerned is approximated locally. In the Nadaraya-Watson estimator, each value of the function is expressed as the weighted average of the dependent variables located in the neighborhood of a given set of explanatory variables. This procedure is basically identical to the local regression of the dependent variable on just a constant term by weighted least squares. The procedure of the local polynomial estimator, on the other hand, constitutes the local approximation of neighbor observations for the dependent variable by not only a constant term, but also a polynomial of explanatory variables with the weight of kernel functions.

For more detailed treatments, see such monographs as Wand and Jones [22].
Similar to the nonparametric estimation procedure of Aït-Sahalia and Lo [2], we characterize call option pricing as the function of two state variables: an inverse moneyness \((M \equiv X/F)\) and maturity \((\tau)\).

Each call contract of the sample is indexed by \(i\) with \(i = 1, \ldots, n\), while \(z_i\) is defined as \((M_i, \tau_i)\). For the \(i\)th contract, we obtain normalized call prices \(H_i^*\) as the dependent variable, and \(z_i\) as the vector of the explanatory variables.

At a point \(z_0\), the Nadaraya-Watson estimator of the call price function \(\widetilde{H}_{NW}(z_0)\) is specified as:

\[
\widetilde{H}_{NW}(z_0) = \sum_{i=1}^{n} [K_h(z_i - z_0)H_i^*],
\]

where \(K_h(u)\) is a kernel function on \(\mathbb{R}^3\) with a diagonal bandwidth matrix \(h\). Each element of the diagonal matrix, \(h_X\), \(h_F\), or \(h_\tau\) is the corresponding bandwidth value. More specifically, the three-dimensional kernel function \(K_h(u)\) is defined as follows:

\[
\sum_{i=1}^{n} [K_h(z_i - z_0)H_i^*] = \frac{\sum_{i=1}^{n} [k_{h_M}(M_i - M_0)k_{h_\tau}(\tau_i - \tau_0)H_i^*]}{\sum_{i=1}^{n} [k_{h_M}(M_i - M_0)k_{h_\tau}(\tau_i - \tau_0)]},
\]

where \(k_{h_M}(u)\), and \(k_{h_\tau}(u)\) are the corresponding kernel functions with the bandwidth values \(h_M\) and \(h_\tau\) respectively.

The main idea of the Nadaraya-Watson estimator is the price of a call contract given the vector of explanatory variables \(z_0\) is approximated by the weighted average of \(H_i^*\) located in the neighborhood of \(z_0\). Several points may be made about the above Nadaraya-Watson
estimator. First, the optimal bandwidth is chosen with the cross-validation method\(^4\). Second, the order of a kernel function is set on the balance between biases and smoothness. Following Aït-Sahalia and Lo [2]’s procedure, we use a fourth-order kernel for \( \tau_0 \), and a second-order Gaussian kernel for \( M_0 \). We follow their choice of kernels when the Nadaraya-Watson estimator is adopted in our estimation. Third, given the nonparametrically estimated \( \tilde{H}_{NW}(z_0) \), we calculate numerically its second derivative with respect to exercise prices.

By using a set of parameters \( \beta_{k_1,k_2}(z_0) \), on the other hand, the local polynomial estimation of the call price function \( H_{LP}(z_0) \) and their derivatives are based on the polynomial defined as:

\[
H_p(z \mid z_0) = \sum_{0 \leq |k| \leq p} \beta_{k_1,k_2}(z_0) \, (z - z_0)^k,
\]

where \( p \) is the degree of the polynomial function, \( z^k \equiv M^{k_1} \tau^{k_2} \), and \( \sum_{0 \leq |k| \leq p} \equiv \sum_{j=0}^{p} \sum_{k_1=0}^{k} \sum_{k_2=0}^{j} \) with \( k_1 + k_2 = j \). More concretely, in the case of \( p = 3 \), \( H_3(z \mid z_0) \) is expressed as:

\[
\begin{align*}
\beta_{00}(z_0) + \beta_{10}(z_0) \, (M - M_0) + \beta_{01}(z_0) \, (\tau - \tau_0) + \\
+ \beta_{20}(z_0) \, (M - M_0)^2 + \beta_{11}(z_0) \, (M - M_0)(\tau - \tau_0) + \beta_{02}(z_0) \, (\tau - \tau_0)^2 + \\
+ \beta_{30}(z_0) \, (M - M_0)^3 + \beta_{21}(z_0) \, (M - M_0)^2(\tau - \tau_0) + \beta_{12}(z_0) \, (M - M_0)(\tau - \tau_0)^2 + \beta_{03}(z_0) \, (\tau - \tau_0)^3
\end{align*}
\]

Then, a set of parameters \( \beta_{k_1,k_2}(z_0) \) is estimated from the following multivariate weighted least

\(^4\)In this paper, following Aït-Sahalia and Lo [2], we apply the cross-validation method to the original function \( \tilde{H}_{NW} \), not to its second derivative. The optimal bandwidth for nonparametrically estimated derivatives is usually hard to obtain: see Stoker [21].
squares ( \( \beta (z_0) = \{\beta_{k_1k_2}(z_0)\} \)):

\[
\min_{\beta(z_0)} \sum_{i=1}^{n} \left[ \left\{ k_{M_1}(M_i - M_0) k_{\tau_1}(\tau_i - \tau_0) \right\} \left\{ H^*_i - \sum_{0 \leq |k| \leq p} \beta_{k_1k_2}(z_0) (z_i - z_0)^k \right\} \right]^2,
\]

where the weight is based on the product of the three kernel functions assigned for each determinant. Given the estimated parameter \( \hat{\beta}(z_0) \), the call price evaluated at \( z = z_0 \) is defined as an estimated constant term, or \( \hat{H}_{LP}(z_0) = \hat{\beta}_{00}(z_0) \). Notice that all terms other than a constant term are dropped at \( z = z_0 \).

If the neighborhood observations \( H^*_i \)'s are regressed on only a constant term, or \( p = 0 \), then we obtain

\[
\hat{H}_0(z_0) = \hat{\beta}_{00}(z_0) = \frac{\sum_{i=1}^{n} [k_{M_1}(M_i - M_0) k_{\tau_1}(\tau_i - \tau_0) H^*_i]}{\sum_{i=1}^{n} [k_{M_1}(M_i - M_0) k_{\tau_1}(\tau_i - \tau_0)]} = \hat{H}_{NW}(z_0).
\]

That is, the Nadaraya-Watson estimator that approximates a function by the weighted average is a special case of the local polynomial estimator. Thanks to more sophisticated local approximation, the local polynomial estimator offers several important advantages as discussed below and in the next subsection.

We have several comments to make about the above local polynomial estimator. First, the degree of the polynomial function \( p \) should be chosen on the balance between biases and smoothness. Ruppert and Wand [20] recommend that the degree of the polynomial function minus the order of the derivative of concern to researchers should be odd. Because of our interest in the second derivative of the call pricing function \( H_{LP}(z \mid z_0) \), \( p \) is set at three for our estimation.
Second, fairly conveniently, the estimator of the second derivative of $\tilde{H}_{LP}(z \mid z_0)$ with respect to the exercise price corresponds to the estimated coefficient on $(M_i - M_0)^2$ multiplied by two and divided by the present future price $F_0$, or $2 \times \beta_{20}(z_0) / F_0$. Notice that all terms other than $\beta_{20}(z_0)$ are dropped when the second derivative with respect to exercise prices is evaluated at $z = z_0$. In other words, the information about marginal effects is summarized in the estimated parameter $\tilde{\beta}_{k_1,k_2}(z_0)$. Hence, in contrast with the Nadaraya-Watson estimator, we do not have to take numerical derivatives in deriving state-price density functions.

Third, the choice of the order of a kernel function is inconsequential in the local polynomial estimator, and the Gaussian kernel is chosen for our estimation. Fourth, the optimal bandwidth $h$ is again chosen based on the cross-validation method.

3.4. The advantage of the local polynomial estimator As the above discussion suggests, the Nadaraya-Watson estimator approximates locally each point of the pricing function by only a constant term. This estimator behaves quite well in general as long as a number of observations are evenly distributed with respect to each explanatory variable. More precisely, it requires plenty of observations on both sides of every point of explanatory variables. As shown by the Monte Carlo simulation conducted by Aït-Sahalia and Lo, the Nadaraya-Watson estimator indeed works fairly properly for their large one-year sample with about 14,400 observations.

5 Note that our regression analysis is based on the following relationship, $H^*_t = H(M_t, \tau_t) + \text{error}$. As stated earlier, the state price density is given as the second derivative of the option pricing function, not the normalized option pricing function, with respect to the exercise price. The second derivative of the option pricing function $F \cdot H(M, \tau)$ with respect to the exercise price is calculated as the second derivative with respect to the inverse moneyness while fixing the present level of the future price,

$$
F \cdot \frac{\partial^2 H(M, \tau)}{\partial X^2} = \frac{1}{F} \frac{\partial^2 H(X/F, \tau)}{\partial M^2}.
$$

6 As in the Nadaraya-Watson estimator, the cross-validation method is applied to the original function $\tilde{H}_{LP}$, not to its second derivative.
The Nadaraya-Watson estimator, however, yields serious biases when there are poor observations on either side of each point of explanatory variables. Such a case always arises near boundaries for any sample; there is no observation on one side at the boundary. As Fan and Gijbels [9] note, there is no practical way to correct biases near boundaries for this estimator.

For the same reason, the Nadaraya-Watson estimator is not suitable for the case where observations are unevenly distributed or highly clustered in the interior of the domain of each explanatory variable. For such a data set, there are many parts where the number of observations is extremely small on either side of a certain point even in the interior. In particular, the numerical estimate of derivatives of an approximated function is biased quite seriously for highly clustered data sets.

The local polynomial estimator, on the other hand, performs well both near boundaries and for sparse interiors. A remarkable feature of this estimator is that the order of the bias is the same for all points, boundary and sparse interior, so long as the degree of the polynomial function is selected appropriately (e.g. Masry [14], Ruppert and Wand [20], and Wand and Jones [22]). As Fan [8] points out, the local polynomial estimator is design-adaptive in that the order of the asymptotic bias does not depend heavily on how unevenly observations are distributed against explanatory variables.

Fortunately, this design-adaptive feature for boundaries and sparse interiors is also applicable to the derivative estimation based on the local polynomial estimator. As mentioned before, the estimates of derivatives are immediately available from the estimated coefficients of a polynomial.

3.5. Stability tests We use a test of stability to identify a structural break in state price densities. More concretely, fixing the sample period (as described in the next section, between January and July/August in 2003 in our case), we split the whole sample into two sub-samples
by a particular date. Then, we apply a test of stability to these consecutive sub samples.

Dette and Neumeyer [6] discuss some procedures for testing the equality of several regression curves estimated by nonparametric method. We slightly modify one of their procedures for the case when comparing nonparametrically-estimated regression derivatives from two consecutive sub-samples.

Let denote the first (second) sub-sample of size $n_1 (n_2)$ as $\{H_{1,i}^*, z_{1,i}\}_{i=1}^{n_1}$ ($\{H_{2,i}^*, z_{2,i}\}_{i=1}^{n_2}$). The covariate $z_1 (z_1)$ has the density function $f_1(z)$ ($f_2(z)$). The test statistic is defined as follows,

$$T_n = \frac{1}{n} \left\{ \sum_{i=1}^{n_1} \left( \hat{b}(z_{1,i}) - \hat{b}_1(z_{1,i}) \right)^2 \pi(z_{1,i}) + \sum_{i=1}^{n_2} \left( \hat{b}(z_{2,i}) - \hat{b}_2(z_{2,i}) \right)^2 \pi(z_{2,i}) \right\}, n \equiv n_1 + n_2,$$

where $\hat{b}_1(z)$ ($\hat{b}_2(z)$) is the state-price density (the second partial derivative of the option pricing function with respect to the exercise price) estimator from the first (second) sub-sample, and $\hat{b}(z)$ is from the whole sample (both sub-samples). We use the indicator function as the weighting function $\pi(z)$, which puts zero-weight if the covariate $z$ is not in the common support of $f_1(z)$ and $f_2(z)$. Without loss of generality, we assume that the first element of the covariate vector corresponds to the exercise price.

The asymptotic distribution under the null hypothesis, the state price densities are stable within the both sub-samples, is simply,

$$n(h_1^4 \prod_{j=1}^{d} h_j^{1/2})(T_n - \mu) \overset{D}{\to} N(0, \Omega)$$

where $d$ is the dimension of the covariate vector $z$, and the bandwidths, $\{h_j\}_{j=1}^{d}$, are the ones used for estimating $\hat{b}(z)$. 
If we use the same bandwidths for all state-price densities, the bias term, \( \mu \), and the variance term, \( \Omega \), are the same as Dette and Neumeyer [6]’s equation (2.16) and (2.17). But, since we use the local polynomial method, the kernel functions in Dette and Neumeyer [6]’s results must be replaced for our test statistic as,

\[
K(u) = e_2 M^{-1} (u^{Q_3})' \prod_{j=1}^{d} K(u_j)
\]

where \( K(u) \) is the usual Gaussian kernel function, The matrix \( M \) is the same as (3.15) in Masry [15], whose elements are the multinomial moments of \( K(u) \) and the polynomial order is set to three. And the vector \( u^{Q_3} \) is the row vector representing the whole polynomial up to three from the lowest to the highest (see APPENDIX in Heckman, Ichimura, and Todd [12]). In empirical research, different bandwidths are used for estimating \( \hat{b}_1(z) \), \( \hat{b}_2(z) \) and \( \hat{b}(z) \). Then the asymptotic bias and variance depend on the ratio of the different bandwidths, and are of the complicated forms.

Dette and Neumeyer [6] recommend the use of the wild bootstrap procedure in implementing the test for its excellent finite sample properties. Therefore we also follow it, and calculate each bootstrapped p-value of the test statistic from 100 bootstrap samples. The results are discussed in the next section.

4. Simulation and Estimation Results

4.1. Simulation results The property of the local polynomial estimator, described in the previous section, makes it suitable for our research purpose. Our data set of option prices is fairly non-uniformly distributed, in particular with respect to moneyness, because the number of
exercise price points is rather limited. As the preceding argument suggests, in such an empirical environment, the local polynomial estimator is expected to perform better than the Nadaraya-Watson estimator.

The observations of option contracts are indeed not distributed uniformly. Figure 5.1 and 5.2 report the distribution of observations of option contracts with respect to both inverse moneyness and maturities; the sample period of these figures is between January and July, 2003. From these figures, it is seen that option contract observations are non-uniformly distributed, and the limited number of listed contracts are reflected in the several spike bars of Figure 5.1.

To illuminate the advantage of the local polynomial estimator, we conduct the following simulation study using the about 10000 observations available from the sample period between January and December in 2003.

We prepare for three sets of simulated data for both non-uniform and uniform designs, whose sample size is 500, 2000, and 10000 respectively. For the non-uniform design, we collect option contract characteristics randomly resampled from the actual observations in 2003 as mentioned above. For the uniform design, we construct a dataset whose observations are distributed uniformly with respect to the observed range of both exercise prices (rather than inverse moneyness) and maturities.

Assuming that the volatility parameter is set to 0.5, call options prices are generated for both designs according to the Black-Scholes/Merton option pricing formula. We add to each generated sample a normally-distributed random component whose standard error is equal to five percent of a true option price. Then, we apply the two estimation methods, the local polynomial estimator and the Nadaraya-Watson estimator to those simulated data according to the two designs. Typical estimation results from both simulation settings are drawn in Figure 6.1 and 6.2.
We choose a bandwidth value as follows. Forty equally-separated points are picked up on the 10% to 90% range of the observed exercise prices. Given the sample average as a futures price, a bandwidth is chosen such that the mean of absolute difference between the true (Black-Scholes/Merton) and estimated densities may be minimized at these forty points of exercise prices.\footnote{The simulation result does not depend on the number of points of exercise prices.} We continue to use the bandwidth value obtained in the first round of simulation for the other rounds. The performance of the two estimation methods is evaluated in terms of the average of the above mean absolute deviation available from each round of simulation. We repeat such simulation procedures one thousand times.

Table 1 reports the performance measures for both uniform and non-uniform designs. We have several remarks on these simulation results. First, even in the case of the uniform design, when the number of data is 500 (comparable to our monthly sample sizes), the local polynomial estimator dominates the Nadaraya-Watson estimator in terms of the average of mean absolute deviations. In other words, the local polynomial estimator bear a reasonable small sample property for the uniform designed data. As the number of data increases, however, there is no substantial difference in the performance between the two. Second, in the case of the non-uniform design, regardless of the number of data, the local polynomial estimator dominates the Nadaraya-Watson estimator in terms of this performance measure. That is, the local polynomial estimator has good properties for both small and large samples under the non-uniform design, which we often encounter in empirical studies.

4.2. \textbf{Structural breaks in state price densities} This subsection presents an application of the local polynomial estimator to the actual option pricing observed during 2003. As discussed
in the introduction, although the stock price index stagnated until May, it had moved upward since June. Figures 7.1 and 7.2 plot the estimated state price density for the sample period between January and April, and between June and July, given the sample average as a futures price. Figure 8 again draws these estimated densities together. These figures demonstrate that the state price density moved substantially around May, 2003.

Following the method described in the previous section, we conduct tests of a structural break by fixing the sample period between January and July. The number of observations is 5013. For this purpose, we divide each month into the first ten days, the mid ten days, and the last ten days.

Table 2 reports test results. In the case of the sample period between January and July, the p-value is found to be the lowest at the break point between last May and first June, or between last May and first June (0%). In the period between January and August, the p-value is the lowest at the same break point (13%). According to these test results, the state-price density changed significantly around mid May or early June.

5. Conclusion Following the method proposed by Aît-Sahalia and Lo [2], this paper conducts a nonparametric estimation of state-price densities implicit in the Osaka stock index option prices. The paper first explores estimation methods suitable for our case where observations are sparse near the boundaries and unevenly distributed in the interior, and state-price densities are not so stable over sub-samples. We find that the local polynomial estimator performs better for data sets as ours than the Nadaraya-Watson estimator. In particular, the local polynomial estimator possesses a great advantage when one is interested in estimating derivatives of approximated functions.
The paper then presents an application of the local polynomial estimator to the option prices observed during 2003 when there was a dramatic change in stock pricing behavior. The estimation result based on the local polynomial estimator finds a reasonably a structural break point in state price densities.

Given our above findings, the local polynomial estimator bears potential applicability in estimating state-price densities, when one has to estimate densities more precisely from sparse data or the instability of functions prevents the pooling of observations for longer sample periods.
REFERENCES


Table 1: Average of Mean Absolute Deviation from True State Price Density

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<th>Uniform Design</th>
<th>local polynomial</th>
<th>Nadaraya-Watson</th>
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<tr>
<td>number of data</td>
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<td>2000</td>
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<table>
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<th>Nadaraya-Watson</th>
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</thead>
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</tr>
<tr>
<td>500</td>
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Table 2: Tests for structural break points

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<th>break point</th>
<th>total observation</th>
<th>first half</th>
<th>second half</th>
<th>p-value of test statistics</th>
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</thead>
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<tr>
<td>Last April/First May</td>
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<td>2643</td>
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<td>3051</td>
<td>1962</td>
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<td>Last May/First June</td>
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<td>3653</td>
<td>1360</td>
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<tr>
<td>Mid June/Last June</td>
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<td>3963</td>
<td>1050</td>
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<td>5013</td>
<td>4195</td>
<td>818</td>
<td>0.19</td>
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Figure 1: Nikkei 225 Stock Index During 2003
(unit: yen)
Figure 2-1: The Volume of Call Contracts, 1993
Figure 2-2: The Volume of Put Contracts

The graph shows the volume of put contracts as a function of moneyness (S/X), where S is the asset price and X is the strike price. The x-axis represents the moneyness ranging from 0.2 to 1.8, while the y-axis represents the volume ranging from 0 to 12000.
Figure 3: Errors in the Parity Condition

Panel 1: Parity Errors (all), 92:2nd

Panel 2: Parity Errors (more than 50 contexts), 92:2nd
Figure 4: Estimated Implied Volatility

Panel 1: Term Structure of Implied Volatility, Call Option, 92:2nd

Panel 2: Term Structure of Implied Volatility, Put Option, 92:2nd
Figure 5.1: Distribution of Option Contracts

Figure 5.2: Distribution of Option Contracts
Figure 8: Comparison of State-Price Densities (t=15), 2003