The stochastic extended path approach

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The views expressed herein are ours and do not necessarily represent the views of Bank of France.

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Motivation

- Severe nonlinearities play sometimes an important role in macroeconomics.
- In particular occasionally binding constraints: irreversible investment, borrowing constraint, ZLB.
- Usual local approximation techniques don't work when there are kinks.
- Deterministic, perfect forward, models can be solved with much greater accuracy than stochastic ones.
- The extended path approach aims to keep the ability of deterministic methods to provide accurate account of nonlinearities.

Outline

1. Extended path approach
2. Stochastic extended path approach
3. Analytical application to Burnside (1998) model
4. Computational aspects of stochastic extended path approach
5. Illustration: a RBC model with irreversible investment

Solving deterministic models

- Perfect foresight, equilibrium, models, after a shock, return asymptotically to equilibirum.
- For a long enough simulation, one can consider that for all practical purpose the system is back to equilibrium.
- This suggests to solve a two value boundary problem with initial conditions for some variables (backward looking) and terminal conditions for others (forward looking).
- In practice, one can use a Newton method to the equations of the model stacked over all periods of the simulation.
- The Jacobian matrix of the stacked system is very sparse and this characteristic must be used to write a practical algorithm.

The extended path approach

Already proposed by Fair and Taylor (1983).

The extended path approach creates a stochastic simulation as if only the shocks of the current period were random.

Solving

$$\mathbb{E}_t \left\{ f(y_{t+1}, y_t, y_{t-1}, u_t) \right\} = 0$$

with auxiliary perfect foresight model

$$f(x_{\tau+1}, x_\tau, x_{\tau-1}, v_\tau) = 0$$

and

$$v_1 = u_t$$
$$v_\tau = 0 \qquad \tau > 1$$

Extended path algorithm

Algorithm 1 Extended path algorithm

1. $H \leftarrow$ Set the horizon of the perfect foresight models.
2. $x^\star$ Compute steady state of the model
3. $y_0 \leftarrow$ Choose an initial condition for the state variables.
4. **for** $t = 1$ to T **do**
5. $\quad x_0 \leftarrow y_{t-1}$ Set initial conditions of perfect foresight model.
6. $\quad u_t \leftarrow$ Draw random shocks for the current period.
7. $\quad (x_1, \dots, x_{T+H}) \leftarrow$ Solve a perfect foresight model.
8. $\quad y_t \leftarrow x_1$ Store simulation at current period.
9. **end for**

Discussion

The extended path approach takes full account of the deterministic nonlinearities of the model.

It neglects the asymmetric effects of future shocks (Jensen's inequality):

$$E_t \{f(u_{t+k})\} \neq f(E_t \{u_{t+k}\})$$

The stochastic extended path approach

The *k*-order stochastic extended path approach takes into account the stochastic nature of the shocks in a few periods ahead

Algorithm 2 First order stochastic extended path algorithm sketch

1. Choose an initial condition for the state variables.
2. **for** $t = 1$ to T **do**
3. Set initial conditions of perfect foresight model.
4. Draw random shocks for the current period.
5. Compute the conditional expectation appearing for the first k periods
6. Solve the large perfect foresight model.
7. $y_t \leftarrow x_{1,1}$ Store simulation at current period.
8. **end for**

Discussion

The extended path approach takes full account of the deterministic nonlinearities of the model.

It takes into account the asymmetric effects of future shocks k-period ahead.

It neglects the asymmetric effects of future shocks in the long run. In most models this effect declines with the discount factor.

Burnside (1998) model

- A representative household
- A single perishable consumption good produced by a single 'tree'.
- Household can hold equity to transfer consumption from one period to the next
- Household's intertemporal utility is given by

$$E_t \left\{ \sum_{\tau=0}^{\infty} \beta^{-\tau} \frac{c_{t+\tau}^{\theta}}{\theta} \right\} \qquad \text{with } \theta \in (-\infty, 0) \cup (0, 1]$$

- Budget constraint is

$$p_t e_{t+1} + c_t = (p_t + d_t) e_t$$

- Dividends d_t are growing at rate x_t;

$$d_t = \exp(x_t) d_{t-1}$$
$$x_t = (1-\rho)\bar{x} + \rho x_{t-1} + \epsilon_t$$

Dynamics

$$
\begin{aligned}
y_t &= \beta E_t \left\{ \exp\left(\theta x_{t+1}\right)\left(1 + y_{t+1}\right)\right\} \\
x_t &= (1-\rho)\bar{x} + \rho x_{t-1} + \epsilon_t
\end{aligned}
$$

where $y_t = p_t / d_t$ is the price-dividend ratio.
It is easy to show that y_t can be written as the current value of future dividends growth rates:

$$
\begin{aligned}
y_t &= E_t \left\{ \sum_{i=1}^{\infty} \beta^{\tau} \exp\left(\sum_{j=1}^{i} \theta x_{t+j} \right) \right\} \\
&= E_t \left\{ \sum_{i=1}^{\infty} \beta^{\tau} \exp\left(\theta \sum_{j=1}^{i} \bar{x} + \rho^i \hat{x}_t + \sum_{\ell=1}^{j} \rho^{j-\ell} \epsilon_{t+\ell} \right) \right\}
\end{aligned}
$$

with $\hat{x}_t = x_t - \bar{x}$.

The exact solution

Using formulas for the distribution of the log-normal random variable, Burnside (1998) shows that the closed form solution is

$$y_t = \sum_{i=1}^{\infty} \beta^i exp\left(a_i + b_i \hat{x}_t\right)$$

where

$$a_i = \theta \bar{x} i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2}\left(i - 2\rho\frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2}\right)$$

and

$$b_i = \frac{\theta\rho\left(1-\rho^i\right)}{1-\rho}$$

The extended path approach

In the extended path approach, one sets future shocks to their expected value, $E\left\{\epsilon_{t+\ell}\right\} = 0$, $\ell = 1, \ldots, \infty$. The corresponding solution is given by

$$y_t = \sum_{i=1}^{\infty} \beta^i exp\left(a_i + b_i \hat{x}_t\right)$$

where

$$a_i = \theta \bar{x} i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \quad i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2}$$

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where

$$a_i = \theta \bar{x} i$$

and

$$b_i = \frac{\theta\rho \quad 1 - \rho^i}{1 - \rho}$$

Numerical simulation

Calibration

$$\bar{x} = 0.0179$$
$$\rho = -0.139$$
$$\theta = -1.5$$
$$\beta = 0.95$$
$$\sigma = 0.0348$$

The deterministic steady state is equal to 12.3035.
The risky steady state, defined as the fix point in absence of shock this period:

$$\tilde{y} = \sum_{i=1}^{\infty} \beta^i exp \quad \theta \bar{x} i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \quad i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2}$$

is equal to 12.4812.

Comparing expended path and closed-form solution

Difference between expended path approximation, $\hat{y}_t$ and closed-form solution, y_t.

- Using 800 terms to approximate the infinite summation
- Computing over 30000 periods

$$\min\left(y_t - \hat{y}_t\right) = 0.1726$$
$$\max\left(y_t - \hat{y}_t\right) = 0.1820$$

- The effect of future volatility isn't trivial

$$\frac{\tilde{y} - \bar{y}}{\bar{y}} = 0.0144$$

- The effect of future volatility doesn't depend much on the state of the economy.

Stochastic extended path

A k-order stochastic expended path approach computes the conditional expectation taking into accounts the shocks over the next k periods.
The closed formula is

$$y_t = \sum_{i=1}^{\infty} \beta^i exp\left(a_i + b_i \hat{x}_t\right)$$

where

$$a_i = \theta \bar{x} i + \begin{cases} \frac{\theta^2 \sigma^2}{2(1-\rho)^2}\left(i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2}\right) & \text{for} \quad i \leq k \\ \frac{\theta^2 \sigma^2}{2(1-\rho)^2}\left(k - 2\rho \frac{\rho^{i-k}-\rho^i}{1-\rho} + \rho^2 \frac{\rho^{2(i-k)}-\rho^{2i}}{1-\rho^2}\right) & \text{for} \quad i > k \end{cases}$$

and

$$b_i = \frac{\theta\rho \;\; 1-\rho^i}{1-\rho}$$

Quantitative evaluation

What is the ability of the stochastic extended path approach to capture the effect of future volatility?

What part of the difference between the risky steady state and deterministic steady state is captured by different values of k?

Deterministic steady state: 12.3035

Risky steady state: 12.4812

The contribution of k future periods

k	Percentage
1	7.4%
2	14.3%
9	50.0%
30	90.1%
60	99.0%

In such a model, it is extremely costly to give full account of the effects of future volatility with the stochastic extended path approach.

Numerical integration

In current Dynare implementation, we use Gaussian quadrature.

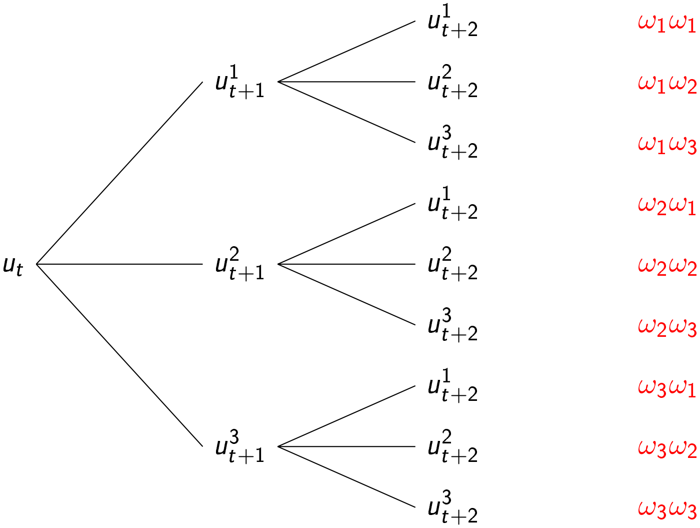
For multiple shocks, we use tensor products (we could replace it by monomial formulas).

The quadrature formula calls for the evaluation of the integrand at a given number of points. Each of these evaluation is the departure point of a separate instance of the extended path algorithm.

When the order of the stochastic extended path algorithm is larger than 1, we use further instances of the stochastic extended path algorithm.

Computation tree

Example for 1 shock, 3 nodes, 2nd order.



Solving a large nonlinear system

For each period of the simulation, one must solve a large nonlinear system, stacking up the equations

- For all variables in one period
- For all periods until approximate return to the steady state
- For all the nodes in the integration step(s)

The Jacobian matrix of this system is very sparse.

The evaluation of the Jacobian matrix can be run in parallel.

Stochastic extended path approach

Algorithm 3 First order stochastic extended path algorithm

$H \leftarrow$ Set the horizon of the perfect foresight models.
$\{(\epsilon_i, \omega_i); i = 1, \dots, p\} \leftarrow$ Compute the weights and nodes of the Gaussian quadrature.
$x^\star$ Compute steady state of the model
$y_0 \leftarrow$ Choose an initial condition for the state variables.
for $t = 1$ to T **do**
    $x_0 \leftarrow y_{t-1}$ Set initial conditions of perfect foresight model.
    $u_t \leftarrow$ Draw random shocks for the current period.
    for $i = 1$ to p **do**
        $v_{2,i} \leftarrow P_i\epsilon$ Build p perfect foresight scenarios.
    end for
    $(x_{1,1}, \dots, x_{T+H,p}) \leftarrow$ Solve the large perfect foresight model.
    $y_t \leftarrow x_{1,1}$ Store simulation at current period.
end for

Irreversible investment

Consider the following RBC model with irreversible investment:

$$\max_{\{c_{t+j},l_{t+j},k_{t+j}\}_{j=0}^{\infty}} \quad \mathcal{W}_t = \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j})$$

$\underline{s.t.}$

$$\begin{aligned}
y_t &= c_t + i_t \\
y_t &= A_t f(k_{t-1}, l_t) \\
k_t &= i_t + (1-\delta)k_{t-1} \\
A_t &= A^\star e^{a_t - \frac{1}{2}\frac{\sigma_\varepsilon^2}{1-\rho^2}} \\
a_t &= \rho a_{t-1} + \varepsilon_t \\
i_t &\geq 0
\end{aligned}$$

Further specifications

The utility function is

$$u(c_t, l_t) = \frac{\left(c_t^{\theta}(1 - l_t)^{1-\theta}\right)^{\tau}}{1 - \tau}$$

and the production function,

$$f(k_{t-1}, l_t) = \left(\alpha k_{t-1}^{\psi} + (1 - \alpha) l_t^{\psi}\right)^{\frac{1}{\psi}}$$

First order conditions

$$
\begin{aligned}
u_c(c_t, l_t) - \mu_t &= \beta \mathbb{E}_t \Big[u_c(c_{t+1}, l_{t+1}) \Big(A_{t+1} f_k(k_t, l_{t+1}) + 1 - \delta \\
&\qquad - \mu_{t+1}(1-\delta) \Big] \\
\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} &= A_t f_l(k_{t-1}, l_t) \\
c_t + k_t &= A_t f(k_{t-1}, l_t) + (1-\delta) k_{t-1} \\
\mu_t \left(k_t - (1-\delta) k_{t-1} \right) &= 0
\end{aligned}
$$

where μ_t is the Lagrange multiplier associated with the constraint on investment.

Calibration

$$\beta = 0.990$$
$$\theta = 0.357$$
$$\tau = 2.000$$
$$\alpha = 0.450$$
$$\psi = -0.500$$
$$\delta = 0.020$$
$$\rho = 0.995$$
$$A^{\star} = 1.000$$
$$\sigma = 0.100$$

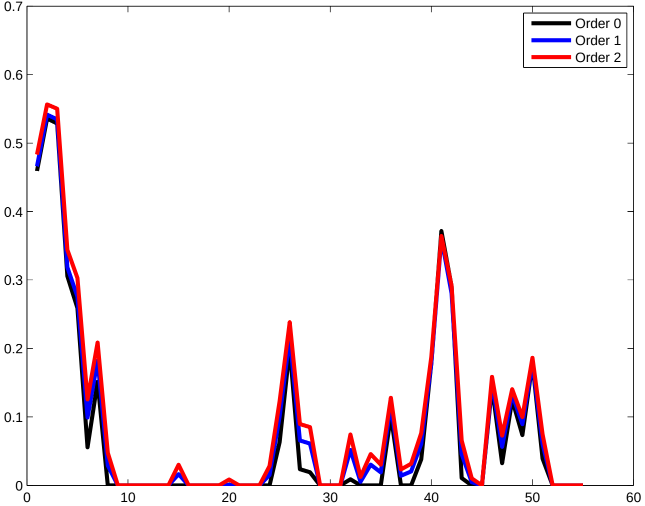
Simulation

Order: 0, 1

Integration nodes: 2

Number of periods for auxiliary simulations: 200

The trajectory of investment



Conclusion and future work

The extended path approach takes into account effects of nonlinearities.

The stochastic extended path approach takes also partially into account nonlinear effects of future volatility.

Possible to use an hybrid approach, using the risky steady state as terminal condition.

The approach suffers from the curse of dimensionality but it can be mitigated by

- using monomial formulas for integration when there are several shocks
- exploiting embarassingly parallel nature of the algorithm