Assessing long run risk in a DSGE model with the stochastic extended path approach
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The views expressed herein are ours and do not necessarily represent the views of Bank of France.

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Introduction

- Finance literature shows that risk premiums vary significantly over time.
- Nonlinear models with expectations, such as DSGE models, have consequences for attitude towards risk and risk premiums.
- Zero lower bound for nominal interest rates (ZLB) affects these mechanisms.
- Hitting the ZLB affects risk premiums. Is the risk of hitting the ZLB in the future priced in asset prices?
A numerical challenge

Perturbation approach:
- 1st order (linearization): no risk premium
- 2nd order: a constant risk premium
- 3rd order: risk premium varies linearly with state variables

Perturbation approach can hardly deal with ZLB.

Value function or policy iterations are impracticable in medium size models.

Global methods such as collocation are harder to implement.

We propose a *hybrid stochastic extended path* approach:
- determinisitic nonlinearities handled with great accuracy,
- Uncertainty in near future handled via quadrature,
- Long run uncertainty captured via perturbation.
The Rudebusch and Swansom (2008) model

Standard DSGE model,
Epstein-Zin preferences to disentangle risk aversion and inter temporal substitution,
firm specific, fixed, capital (no accumulation),
Calvo pricing
Taylor rule
Epstein-Zin preferences

Welfare:

\[ W_t = u(c_t, l_t) + \beta \left( \mathbb{E}_t W_{t+1}^{1-\alpha} \right)^\frac{1}{1-\alpha} \]

if \( u(c_t, l_t) > 0 \) everywhere, and by:

\[ W_t = u(c_t, l_t) - \beta \left( \mathbb{E}_t (-W_{t+1})^{1-\alpha} \right)^\frac{1}{1-\alpha} \]

if \( u(c_t, l_t) > 0 \) everywhere.

Period utility:

\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \chi_0 \frac{l_t^{1+\chi}}{1+\chi}. \]

Household’s resource constraint

\[ p_t a_t + P_t c_t = w_t l_t + d_t + p_t a_{t-1} \]
First order conditions:

\[
\frac{(1 - \chi) \ell_t^X}{(1 - \gamma) c_t^{-\gamma}} = \frac{w_t}{P_t}
\]

\[
c_t^{-\gamma} = \beta \mathbb{E}_t \mathbb{E}_t V_{t+1}^{1-\alpha} \frac{\alpha}{1-\alpha} V_{t+1}^{-\alpha} c_{t+1}^{-\gamma} (1 + r_{t+1}) \frac{P_t}{P_{t+1}}
\]

Stochastic discount factor:

\[
m_{t+1} = \left( \frac{V_{t+1}}{\mathbb{E}_t V_{t+1}^{1-\alpha} \frac{1}{1-\alpha}} \right)^\alpha \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \frac{P_t}{P_{t+1}}.
\]
Firms

Production function

\[ y_t(i) = A_t^{1-\eta} l_t(i)^\eta \]

with

\[ \log A_t = \rho_A \log A_{t-1} + \epsilon_t^A, \]

Calvo pricing optimal price:

\[ p_t(i) = \frac{(1 + \theta) \mathbb{E}_t \sum_{j=0}^{\infty} \xi_j}{\mathbb{E}_t \sum_{j=0}^{\infty} \xi_j} \frac{t, t+j m_{c{t+j}(i)} y_{t+j}(i)}{t, t+j y_{t+j}(i)}, \]

where marginal cost, \( m_{c_t} \), is:

\[ m_{c_t}(i) = \frac{w_t l_t(i)}{\eta y_t(i)}. \]
Monetary policy

Taylor rule

\[ i_t = \rho_i i_{t-1} + (1 - \rho_i) \left[ \frac{1}{\beta} + \bar{\pi}_t + g_y \frac{Y_t - \bar{Y}}{\bar{Y}} - g_\pi (\bar{\pi}_t - \pi^*) \right] + \varepsilon_t \]

with ZLB:

\[ i_t = \max \left( 0, \rho_i i_{t-1} + (1 - \rho_i) \left[ \frac{1}{\beta} + \bar{\pi}_t + g_y \frac{Y_t - \bar{Y}}{\bar{Y}} - g_\pi (\bar{\pi}_t - \pi^*) \right] \right) \]
Long term bond

A default-free nominal console which pays a geometrically declining coupon in every period to perpetuity:

\[
\widetilde{p}_t^{(n)} = 1 + \delta_c E_t \cdot t+1 p_{t+1}^{(n)}
\]

where \( \delta_c \) is the rate of decay of the console’s coupon.

The risk-neutral price of the console, \( \hat{p}_t^{(n)} \), is:

\[
\hat{p}_t^{(n)} = E_t \sum_{j=0}^{\infty} e^{-i_{t,t+j} \delta_c}
\]

with \( i_{t,t+j} = \sum_{n=0}^{j} i_n \).

The term premium \( \psi_t^{(n)} \) is defined as:

\[
\psi_t^{(n)} = \log \frac{\delta_c \hat{p}_t^{(n)}}{\hat{p}_t^{(n)} - 1} - \log \frac{\delta_c \hat{p}_t^{(n)}}{\hat{p}_t^{(n)} - 1}
\]
Hybrid stochastic extend path approach

Solving deterministic models,
Using quadrature to take into account near future uncertainty,
Using perturbation for long run uncertainty.
Solving deterministic model

Slight approximation: system converges in finite time, $H$.

Solving stacked system by Newton-type method:

$$
\begin{align*}
\begin{cases}
  f(y_0, y_1, y_2, u_1) &= 0 \\
  f(y_1, y_2, y_3, u_2) &= 0 \\
    &\vdots \\
  f(y_{H-1}, y_H, y_{H+1}, u_T) &= 0 
\end{cases}
\end{align*}
$$

with $y_0$ given, and $y_{H+1} = \bar{y}$. 
Extended path algorithm

Suggested by [?]:

Algorithm 1 Extended path algorithm

1. $H \leftarrow$ Choose the horizon of the perfect foresight models.
2. $y_0 \leftarrow$ Choose an initial condition for the state variables.
3. for $t = 1$ to $T$ do
4. \hspace{1em} $v_t \leftarrow$ Draw independent uniform variates ($n_s \times 1$).
5. \hspace{1em} $u_t \leftarrow P^{-1} v$, where $P^{-1} P = Q$, the covariance matrix of shocks.
6. \hspace{1em} $y_t \leftarrow$ Solve a perfect foresight model with terminal condition $y_{t+H+1} = y^*$.
7. end for

can’t give account of risk premium because of certainty equivalence assumption.
Stochastic extended path algorithm

In the first $K$ periods, the conditional expectation is computed by quadrature:

$$\sum_{i=1}^{P} \omega_i f(y_{j-1,t-1}, y_j, t, y_{i|j,t+1}, u_t) = 0, \text{ for } t = 1, \ldots, K \text{ and } j = 1, \ldots, J_t$$

$$f(y_{j,t-1}, y_j, t, y_{j,t+1}, 0) = 0, \text{ for } t = K + 1, \ldots, H \text{ and } j = 1, \ldots, J_K$$
Unscented transformation (I)

Unscented transformations have been proposed by Julier and al. to approximate mean and variance of nonlinear functions of random variables.

An attractive and economical approach to integrate in $R^m$ is to use a formula with $2m+1$ nodes.

Conditional expectation $\mathbb{E}_t f(y(u_{t+1}), y_t, y_{t-1}, u_t)$ can be approximated by $\sum_{i=0}^{2m} \omega_i f(y(e_i), y_t, y_{t-1}, u_t)$. 
Unscented transformation (II)

If the vector of exogenous random shocks $u_t$ follows a multivariate Gaussian distribution with mean 0 and covariance matrix $\Sigma_u$, then, with $P'P = \Sigma_u$, the nodes $e_i$, $i = 0, \ldots, 2$ are computed as follows:

$$e_0 = 0$$

$$e_{2i-1} = \sqrt{e_{2i}^2 + \alpha P_i} \quad i = 1, \ldots,$$

$$e_{2i} = -\sqrt{e_{2i}^2 + \alpha P_i} \quad i = 1, \ldots,$$

The weights $\omega_i$ is:

$$\omega_0 = \frac{\alpha}{\alpha + \alpha}$$

$$\omega_i = \frac{1}{2(\alpha + \alpha)} \quad i = 1, \ldots, 2$$

$$\sum_{i=0}^{2m} \omega_i = 1.$$  

The unscented transformation let us recover the mean and the covariance matrix exactly for any 3rd order polynomial function.
Stochastic extended path of higher order

Figure: Paths of future innovations in a sparse tree. Illustration with 2 shocks, 5 nodes and order 3
Hybrid stochastic extended path

It would take a very high order of stochastic extended path to fully take into account the effects of future volatility. Let’s introduce $\sigma$, the stochastic scale of the model, such that

$$u_{t+1} = \sigma \varepsilon_t$$

where $\varepsilon_t$ is a vector of auxiliary random variables. Consider the vector of solution functions $g()$ such that

$$y_t = g(y_{t-1}, u_t, \sigma)$$

and the original model is satisfied. Functions $g()$ are unknown. The hybrid stochastic extended path approach considers an Taylor expansion in the sole direction of $\sigma$:

$$\mathbb{E}_t f \left( g \left( g \left( y_{t-1}, u_t, \sigma \right), \sigma \varepsilon_{t+1}, \sigma \right), g \left( y_{t-1}, u_t, \sigma \right), y_{t-1}, u_t \right)$$

$$= f \left( g \left( y_{t-1}, u_t, 0 \right), 0, 0 \right), g \left( y_{t-1}, u_t, 0 \right), y_{t-1}, u_t \right)$$

$$+ \mathbb{E}_t \sum_{i=1}^{\infty} \frac{1}{i!} \frac{\partial^i F}{\partial \sigma^i} \sigma^i.$$
Hybrid stochastic extended path at 2nd order

Considering a 2nd order perturbation, the terms in periods $K + 1$, entering into the quadrature formula for period $K$, would be corrected in the following manner:

\[ y_{t+K+1} = y_{t+K+1} + \frac{1}{2}g\sigma^2 \]

where $y_{t+k+1}$ is the value computed by the deterministic simulation approach.

At 3rd order, the correction would be instead

\[ y_{t+K+1} = y_{t+K+1} + \frac{1}{2}g\sigma^2 + \frac{1}{2}g_{y,\sigma^2}y_{t+K} \]
Calibration

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Quantitative experiment

A stochastic simulation of 100 periods of an artificial economy, Only one shock (total factor productivity $A_t$), Unscented transformation with 3 nodes. Stochastic extended path order ranging from 1 to 15 orders. Hybrid correction for long run uncertainty of 2nd order. Due to a spell of rapid technological growth, nominal interest rates fall to 0 in period 20. ZLB episode last about 6 periods.
Nominal interest rate
Term premium

![Graph showing Term premium with two lines representing 'no ZLB' and 'ZLB'.]
Consumption
Bond yield
Term premium at different orders
Conclusions

*Hybrid stochastic extended path* approach let us analyze quantitatively how the ZLB affects asset prices and term premiums.

Effect on impact, but little evidence of large expectation effect, so far.

Next steps:
- check sensitivity to integration formula;
- compare with 3rd order perturbation, in absence of ZLB;
- try to get a measure of accuracy.